ANALYSIS PH.D. QUALIFYING EXAMINATION
August 1998

Measure Theory

1. Let $f_n \in L^1([0,1], m)$ (m is the Lebesgue measure) be a sequence of functions which converges in the L^1-norm to 0. Does it follow that $\frac{f_n}{n} \to 0$ almost everywhere on $[0,1]$?

2. Show that almost all real numbers (with respect to the Lebesgue measure) have the digit 7 in their decimal expansion.

3. Let $\mathcal{M}$ be the collection of all sets E in the unit interval $[0,1]$ such that either E or its complement is at most countable. Show that $\mathcal{M}$ is a σ-algebra. If $g(x) = x$ for $0 \le x \le 1$, show that g is not $\mathcal{M}$-measurable.

4. Let f be nonnegative and Lebesgue integrable in the interval $[0,1]$, and suppose that for every integer $n = 1, 2, \ldots$,

$$\int_0^1 [f(x)]^n \, dx = \int_0^1 f(x) \, dx.$$

Show that f must be a.e. equal to the characteristic function χ_E of some measurable set $E \subset [0,1]$.

Functional Analysis

1. Let A_n be a sequence of bounded operators in a Banach space converging in the norm to an operator A. Let $\lambda \in \mathrm{Spec} A$. Prove that for every $n > 0$ there is a sequence of complex numbers $\lambda_n \in \mathrm{Spec} A_n$ such that $\lambda_n \to \lambda$.

2. Let (X, μ) be a measure space. Fix $f \in L^\infty(X, \mu)$ and consider an operator $M_f : L^2(X, \mu) \to L^2(X, \mu)$ defined by $M_f(g) = fg$. Show that $\|M_f\| \le \|f\|_\infty$. Determine all the functions $f \in L^\infty(X, \mu)$ for which M_f is onto.

3. Show that the operator A defined on $L^2([0,1], m)$ (m is the Lebesgue measure) by

$$(Af)(x) = \int_0^1 xyf(y) \, dy$$

is compact and self-adjoint. Find all its non-zero eigenvalues and the corresponding eigenvectors.

4. Let V be a normed linear space. Prove that for any $v \in V$

$$\|v\| = \sup\{|v^*(v)| : v^* \in V^*, \|v^*\| = 1\},$$

where V^* is the dual space to V.

Complex Analysis

1. Find all entire functions f which satisfy

$$\mathrm{Re} f(x+iy) = x^2 - y^2 + 2$$

for all $x, y \in \mathbb{R}$.

2. Find the image of the domain $\{x+iy : 0 < x < 1,\ y > 0\} \subset \mathbb{C}$ under the map $f(z) = z^{-1}$.

3. Find all poles of the function

$$f(z) = \frac{1-\cos z}{(e^z - 1)^3}$$

4. Consider the equation

$$z + \lambda - e^z = 0,$$

where λ is a real number which is strictly bigger than 1. Show that this equation has a unique root in the left half plane $\mathrm{Re} z < 0$. Show also that this root is a real number.