ANALYSIS QUALIFYING EXAMINATION

AUGUST 22, 1994

There are 4 problems from Complex Analysis, 4 problems from Real Analysis and 4 problems from Functional Analysis. A perfect score will be awarded for doing 2 problems from each section. Partial credit will be awarded, but given a choice between writing something incorrect and writing nothing at all, your grade will suffer less from writing nothing at all.

Complex Analysis

1. Show that if $0 < r < R$ then there exists $\epsilon > 0$ such that

$$\sup\{|p(z) - z^{-1}| : r < |z| < R\} > \epsilon,$$

for every polynomial $p(z)$.

2. Let $a > 1$. Evaluate

$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta}.$$

3. Suppose Ω is a region in the complex plane, $a \in \Omega$ and f is an analytic function on $\Omega \backslash \{a\}$. Prove the Casorati-Weierstrass Theorem: if f has an essential singularity at a then the image under f of every punctured open disk centered at a is dense in $\mathbb{C}$.

4. State and prove Schwarz's Lemma.

Real Analysis

1. If $\mathbf{x} = (x_1, x_2, \dots) \in \ell^1(\mathbb{N})$ then denote by $\|\mathbf{x}\|$ the ℓ^1 norm of $\mathbf{x}$:

$$\|\mathbf{x}\| = \sum_{j=1}^{\infty} |x_j|.$$

In addition, if $\mathbf{y} \in \ell^\infty(\mathbb{N})$ then denote by $\mathbf{x} \cdot \mathbf{y}$ the sum

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{\infty} x_j y_j.$$

Show that if $\{\mathbf{x}_n\}_{n=0}^{\infty} \subset \ell^1(\mathbb{N})$, and if $\lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{y} = 0$ for every $\mathbf{y} \in \ell^\infty(\mathbb{N})$, then $\lim_{n\to\infty} \|\mathbf{x}_n\| = 0$.

2. Show that it is possible to write $\mathbb{R} = A \cup B$, where A is a Borel set of zero Lebesgue measure, and B is of first category.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

3. Let $\{f_n\}$ be a bounded sequence in $L^1(0,1)$. Suppose that $\lim_{n\to\infty} \int_0^1 f_n g$ exists for every $g \in C([0,1])$.

(a) Prove that there exists a Borel measure γ such that

$$\int_0^1 g d\gamma = \lim_{n\to\infty} \int_0^1 f_n g, \quad \text{for every } g \in C([0,1]).$$

(b) Is it true that $\lim_{n\to\infty} \int_0^1 f_n g$ exists for every bounded measurable function g?

4. Suppose that $K(x,y)$ is a Lebesgue measurable function on $\mathbb{R}^2$ for which there exists a constant C such that

$$\int |K(x,y)| dy < C, \qquad \text{for almost every } x,$$

and

$$\int |K(x,y)| dx < C, \qquad \text{for almost every } y.$$

Prove that if $f \in L^p(\mathbb{R})(1 < p < \infty)$, then the function $Tf(x) = \int K(x,y) f(y) dy$ lies in $L^p(\mathbb{R})$, and that there exists some constant C' such that

$$\|Tf\|_{L^p} \leq C' \|f\|_{L^p},$$

for every $f \in L^p(\mathbb{R})$. (Hint: Write $|K(x,y)| = |K(x,y)|^\alpha |K(x,y)|^{1-\alpha}$.)

Functional Analysis

1. Let S and T be linear transformations from a Hilbert space $\mathcal{H}$ to itself such that $\langle Sv, w\rangle = \langle v, Tw\rangle$, for all $v, w \in \mathcal{H}$. Prove that S and T are bounded operators.

2. Let $\mathcal{Y}$ be a finite-dimensional subspace of a normed linear space $\mathcal{X}$. Prove that there is a **closed** subspace $\mathcal{Z}$ of $\mathcal{X}$ such that $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$ (that is, every vector in $\mathcal{X}$ my be written, in a unique way, as a sum of one vector in $\mathcal{Y}$ and one vector in $\mathcal{Z}$).

3. Show that every subspace of a normed linear space which is closed in the norm topology is also closed in the weak topology. By considering $c_0(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$, show that not every norm-closed subspace of a dual space is weak*-closed.

4. For $s \in \mathbb{R}$ define an operator $T_s : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$T_s f(x) = f(x+s) \quad (f \in L^2(\mathbb{R})).$$

Show that for every $f \in L^2(\mathbb{R})$ the map $s \mapsto T_s f$ is a continuous function from $\mathbb{R}$ into $L^2(\mathbb{R})$. Show that the map $s \mapsto T_s$ is **not** a continuous function from $\mathbb{R}$ into the space of bounded linear operators on $L^2(\mathbb{R})$, equipped with the operator norm.