ANALYSIS QUALIFYING EXAMINATION

AUGUST 19, 1991

There are 4 problems from Complex Analysis and 8 problems from Real and Functional Analysis. Work at least 2 problems from the Complex Analysis section and at least 4 problems from the Real and Functional Analysis section.

Take care in what you write. Incorrect statements will detract from your score.

Complex Analysis

1. Consider the following statement:

"There exists a sequence $\{f_n\}$ of functions that are analytic on a region containing the closed unit disc, and satisfy
(i) $\lim_{n\to\infty} f_n(0) = 0$ and
(ii) $\lim_{n\to\infty} f_n(z) = 1$ uniformly on the unit circle."

If the statement is true, give an example; otherwise show that it is false.

2. Let $p(z)$ be a polynomial of degree n and write $M(r) = \max_{|z|=r} |p(z)|$. Prove that if

$$\frac{M(r_1)}{r_1^n} = \frac{M(r_2)}{r_2^n}$$

for some fixed $0 < r_1 < r_2$, then $p(z) = cz^n$ for some constant c.

3. Define the Poisson kernel $P_r(\theta)$ and prove that it satisfies the following:

a. $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1$;
b. $P_r(\theta) > 0$ for all θ;
c. $P_r(\theta) < P_r(\delta)$ if $0 < \delta < |\theta| \le \pi$;
d. for each $\delta > 0$, $\lim_{r\to 1^-} P_r(\theta) = 0$ uniformly in θ for $\pi \ge |\theta| \ge \delta$.

4. Let a be a nonzero real number and write

$$f(z) = \frac{\cot(\pi z)}{z^2 + a^2}.$$

a. Where are the poles of f and what is the residue of f at each pole?
b. Use a rectangular contour integral of f to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}.$$

(Hint: $\cot z$ is bounded on any set at positive distance from the real line).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Real and Functional Analysis

1.

a. Give an example of a measure space $(X, \mathcal{M}, \mu)$ and a sequence $\{f_n\}$ of non-negative measurable functions such that

$$\lim_{n\to\infty} f_n(x) = 0 \text{ for all } x, \text{ but } \int f_n d\mu = 1 \text{ for each } n.$$

b. Give an example of a measure space $(X, \mathcal{M}, \mu)$ and a sequence $\{f_n\}$ of non-negative measurable functions such that

$$\lim_{n\to\infty} f_n(x) \text{ does not exist for any } x, \text{ but } \lim_{n\to\infty} \int f_n d\mu = 0.$$

2. Suppose $(X, \mathcal{M}, \mu)$ is a measure space, $1 \le p < \infty$ and $\{f_n\}$ is a sequence in $L^p(\mu)$ that converges to $f \in L^p(\mu)$. Also suppose that $\{g_n\}$ is a sequence in $L^\infty(\mu)$ that converges pointwise to g. Show that if $\|g_n\|_\infty \le 1$ for each n, then $f_n g_n$ converges to fg in $L^p(\mu)$.

3. Let ν be a signed measure on the Borel subsets of the unit interval $I = [0,1]$ such that $|\nu|(I) = 1$, and $\nu(I) = 0$. Suppose that there is a continuous real valued function f on I such that $\|f\|_\infty \le 1$ and

$$\int_I f \, d\nu = 1.$$

Prove that the support of $|\nu|$ is a proper subset of I.

4. Show that if f and g are functions in $L^2(\mathbb{R})$ and h is defined by the formula

$$h(x) = \int_{-\infty}^{\infty} f(y) g(x-y) dy,$$

then h is continuous and

$$\lim_{|x|\to\infty} h(x) = 0.$$

5. Let f_n be a sequence of non-negative continuous functions on $[0,1]$ which is bounded at every point. Prove that this sequence is uniformly bounded on some subinterval of $[0,1]$.

6. Suppose $(X, \mathcal{M}, \mu)$ is a finite measure space and $1 \le r < s < \infty$. Show that there is a constant $K > 0$ such that if $f \in L^s(\mu)$, then

$$\|f\|_r \le K \|f\|_s.$$

7.

a. State Fubini's theorem for non-negative functions.
b. Give an example of real valued functions for which the conclusion of Fubini's theorem is false.
c. Give an example that shows that the conclusion is false if a measure is not σ-finite.

8. Let S denote the (backward) shift operator on $\ell^\infty(\mathbb{N})$, i.e. $(Sx)_j = x_{j+1}$, and write $\mathcal{M}$ for the subspace of elements of the form $x - Sx$.

a. If e denotes the vector with 1 in each entry, then show that

$$dist(e, \mathcal{M}) = 1$$

b. Prove that there is a continuous norm 1 linear functional ϕ on $\ell^\infty(\mathbb{N})$ such that $\phi(Sx) = \phi(x)$ for each $x \in \ell^\infty(\mathbb{N})$.