Ph.D. QUALIFYING EXAMINATION IN ANALYSIS

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To obtain a perfect score, you should give complete solutions to two problems in each section.

Section 1 - Measure Theory

1. Let (X, μ) be a finite measure space. A sequence of measurable functions (f_n) on X is said to *tend to zero in measure* if, for every $\epsilon > 0$, the measure of the set

$$\{x \in X : |f_n(x)| > \epsilon\}$$

tends to zero as $n \to \infty$.

Prove that (f_n) tends to zero in measure if and only if

$$\int_X \frac{|f_n|}{1 + |f_n|} d\mu \to 0 \quad \text{as } n \to \infty.$$

Give an example to show that this result can fail if the measure of X is allowed to be infinite.

2. Let A be a subset of $\mathbb{R}$ having positive Lebesgue measure. Prove that the set

$$A - A := \{a - a' : a, a' \in A\}$$

contains a neighborhood of the origin.

3. Prove that

$$\int_0^r \frac{\sin x}{x}\, dx = \int_0^\infty \int_0^r e^{-xy} \sin x \, dx \, dy.$$

Deduce that as $r \to \infty$, $\displaystyle\int_0^r \frac{\sin x}{x}\, dx \to \frac{\pi}{2}$. Justify all your statements carefully.

4. Let $f, g \in L^1(\mathbb{R})$ (with respect to the Lebesgue measure). Prove that there exist real numbers a_{kn}, x_{kn} such that $\|f * g - h_n\|_{L^1} \to 0$ as $n \to \infty$, where $h_n(x) = \sum_{k=1}^n a_{kn} g(x - x_{kn})$ and $\sum_{k=1}^n |a_{kn}| \le \|f\|_{L^1}$.

Section 2 - Functional Analysis

1. Give an example of a sequence of *unit* vectors in a Hilbert space that converges weakly, but does not converge in norm. Is it possible to find such an example where the limit is also a unit vector? Give reasons.

2. Let e and f be *unit* vectors in a (real) Banach space satisfying

$$\|2e + f\| = \|e - 2f\| = 3.$$

   Show using the Hahn-Banach theorem that there exist continuous linear functionals ϕ and ψ of norm 1 such that

$$\phi(e) = \phi(f) = 1, \quad \psi(e) = -\psi(f) = 1.$$

   Deduce that for any scalars λ and μ, $\|\lambda e + \mu f\| = |\lambda| + |\mu|$.

3. Let T be a bounded operator on a Hilbert space. One says that T is *normal* if $T^*T = TT^*$. Prove that if T is normal and $T^n = 0$ for some positive integer n, then $T = 0$.

4. Let $\mathcal{M}(K)$ denote the set of Borel, regular, complex measures on a compact set K. Prove that $\|\nu\| := |\nu|(K)$ is a norm on $\mathcal{M}(K)$ that makes it a Banach space, without using that $\mathcal{M}(K) = C(K)^*$.

Section 3 - Complex Analysis

1. Suppose $a \in \mathbb{R}$, $|a| \neq 1$. Using residues, evaluate
$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}.$$

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous real-valued function with compact support. Show that the *Fourier transform*
$$\hat{f}(x) = \frac{1}{2\pi} \int e^{-ixt} f(t)dt$$
extends to an entire function on the complex plane.

   Deduce that it is impossible for both f and $\hat{f}$ to have compact support on the real line, unless f is identically zero.

3. Find a conformal mapping of the region
$$\{z : |z \pm i| < \sqrt{2}\}$$
onto the unit disk, as a composition of elementary conformal maps.

4. Let ν be a complex measure on $\mathbb{C}$ with support a compact set $K \subset \mathbb{C}$. Let $f(z) = \int_K (z - w)^{-1} d\nu(w)$. Prove that f is holomorphic at any point $z \notin K$ and find the radius of convergence of f at z.