Ph. D. QUALIFYING EXAMINATION IN ANALYSIS

WEDNESDAY, AUGUST 20, 2003

To get a perfect score you should give complete solutions of two problems from each section.

You may use (correctly) standard results covered in the 501/502 sequence with a reference but without a complete formulation. Examples are Baire category theorem, Monotone Convergence theorem, Schwartz lemma, Riemann Mapping theorem, Hahn-Banach theorem, Spectral theorem for compact self-adjoint operators, an so on. If you have any doubts whether a particular result can be quoted please ask a faculty member proctoring the exam.

SECTION 1

1.1. Consider the set A of all real numbers $x \in [0,1]$ such that for each $n = 0, 1, 2, \dots$ there is an m with $2^n < m \le 2^{n+1}$ such that the m'th digit of a decimal expansion for x is zero. (Formally speaking, $x = \sum 10^{-k} x_k$ with each $x_k \in \{0, 1, \dots, 9\}$ and $x_m = 0$.) Prove that A is Lebesgue measurable and find out whether the measure of A is positive or zero.

1.2. Let f be a real-valued function on $[0,1]$ such that $|f|$ is (Lebesgue) measurable and the set $\{x \in [0,1] : f(x) > 0\}$ is measurable. Prove that f is measurable.

1.3. Let λ denote Lebesgue measure and let $f\colon [0,1] \to [0,1]$ be a differentiable function such that for every Lebesgue measurable set $A \subset [0,1]$ one has $\lambda(f^{-1}(A)) = \lambda(A)$. Prove that either $f(x) = x$ or $f(x) = 1 - x$ (the derivative f' is not assumed to be continuous).

1.4. Let μ be a measure (on some measure space X) and let $f_n,\ n = 1, 2, \dots$ be a sequence of real-valued measurable functions on X. Suppose that for every $\epsilon > 0$, the sum

$$\sum_{n=1}^{\infty} \mu\{x : |f_n(x)| > \epsilon\}$$

is finite. Prove that f_n converges to zero almost everywhere.

SECTION 2

2.1. Let f be an entire function which takes every value at most finitely many times. Prove that f is a polynomial.

2.2. Let $D \subseteq \mathbb{C}$ be a simply connected domain. Suppose that there exists a holomorphic map $f : D \to D$ which is not the identity and has two fixed points. Prove that $D = \mathbb{C}$.

2.3. Show that

$$\int_0^\infty \frac{\log x}{1+x^4}\, dx = -\frac{\pi^2}{8\sqrt{2}}.$$

2.4. Find (explicitly) a conformal mapping of the region

$$|z+i| < \sqrt{2}, \quad |z| > 1$$

onto the unit disk.

SECTION 3

3.1. Let A be a compact linear operator in a Hilbert space. Is A^* compact? Give a proof or a counterexample.

3.2. Let A be a compact normal operator in a complex Hilbert space H. (Recall that *normal* means that $AA^* = A^*A$.) Prove that there is an orthonormal basis for H made up of eigenvectors for A.

3.3. Consider the sequence spaces ℓ^p , $1 \leq p < \infty$. For which values of p is the following property true:

for every continuous linear functional f on ℓ^p there exists $x \in \ell^p$ such that $\|x\| = 1$ and $f(x) = \|f\|$?

3.4. Prove that for any Borel measure μ on the unit interval there exists an extension to a *finitely additive* measure μ^* defined for all subsets of the interval and such that μ^* does not exceed the outer measure defined by μ.