

Analysis
Qualifying Examination.
August 15, 2001

Real Analysis

1. Construct a perfect nowhere dense set on $[0, 1]$ with positive Lebesgue measure.
2. Let $\{f_n\}$ be a sequence of measurable real-valued functions on $[0, 1]$. Show that the set of x for which $\lim_{n \rightarrow \infty} f_n(x)$ exists is measurable.
3. Let $f : \mathbf{R}^n \rightarrow \mathbf{R}$ be a differentiable function, and $df = 0$ for all x on a convex set C . Show that $f = \text{const}$.
4. Prove or disprove: if a sequence of real-valued functions on (a, b) converges to f uniformly, then it converges to f in the sense of L^1 .

Functional Analysis

1. For which continuous functions g is the operator of multiplication by g a bounded operator from $L^2(0,1)$ into $L^2(0,1)$?
2. Let A be a bounded linear operator on a Hilbert space with $\|A\| < 1$. Prove that $(I - A)^{-1}$ exists (here I is the identity operator).
3. Given: $f \in C^\infty([0,1])$ such that $\forall x \exists N$ such that $f^{(n)}(x) = 0 \forall n \geq N$. Prove: f is a polynomial on some subinterval of $[0,1]$.
4. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, \dots$ be a natural basis in l_2 (the space of all square summable sequences of real numbers). Define a sequence of linear operators A_n as follows:

$$A_n \mathbf{e}_k = \begin{cases} \mathbf{e}_1 & \text{if } k = n \\ \mathbf{0} & \text{if } k \neq n. \end{cases}$$

Prove that A_n cconverges weakly to 0 as $n \rightarrow \infty$.

Complex Analysis

1. Prove that if f is an entire function, and if $\operatorname{Re} f(z) \leq M < \infty$ for all z , then f is constant.
2. Prove that $e_n(z) = \sum_{k=0}^n \frac{z^k}{k!}$ has no zeros in $|z| < 100$ for n large enough.
3. Let f be a one-to-one holomorphic function from the unit disk onto itself, with $f(0) = 0$, $f'(0) > 0$. Prove that $f(z) = z$.
4. Map conformally the intersection of two disks $|z-i| \leq \sqrt{2}$ and $|z+i| \leq \sqrt{2}$ onto the unit disk.