MATH 501: REAL AND COMPLEX ANALYSIS

FALL 1996

A.Katok

PROBLEM SET # 1

Part 1. **SET THEORY, METRIC SPACES**

Due on Wednesday 9-4-96

1. The set of all sequences of real numbers has the power of continuum.

2. The set of all continuous functions on the real line has the power of continuum.
Hint: Try to reduce this statement to the previous problem.

3. The set of of all real–valued functions of a real variable has the same power as the set of all subsets of the real line.
Hint: Use the fact that the set of points of the plane has the power of continuum.

4. Given an $\epsilon > 0$ construct an example of an open subset S_ϵ of the real line such that

   a) S_ϵ is dense in $\mathbb{R}$;
   b) the sum of lengths of all intervals comprising S_ϵ is less than ϵ.

5. Write a complete proof of equivalence of two definitions of continuity for a map $f: X \to Y$, where X and Y are metric spaces:

   a) For any open set $A \subset Y$ its pre–image $f^{-1}(A) \subset X$ is open in X;
   b) for any $x \in X$ and any $\epsilon > 0$ there exists a $\delta > 0$ such that if $\text{dist}_X(x, y) < \delta$ then $\text{dist}_Y(fx, fy) < \epsilon$.

6. Write a complete proof of equivalence of two definitions of compactness of a metric space:

   a) Any open cover contains a finite sub–cover;
   b) any sequence of points contains a converging subsequence.

7. Let Γ be a closed convex curve in the plane. Proof that it has a *diameter*, i.e. there are two points $x_1, x_2 \in \Gamma$ such that for any $y_1, y_2 \in \Gamma$, $\text{dist}(y_1, y_2) \leq \text{dist}(x_1, x_2)$.

Part 2. **CANTOR SPACES, RIEMANN INTEGRATION**

Due on Monday 9-9-96

8. Recall that a *homeomorphism* between metric spaces is a one-to-one continuous map whose inverse is continuous.

Let A be a closed bounded subset of the real line which does not contain isolated points or intervals (a perfect nowhere dense set). Construct a homeomorphism between A and the standard middle–third Cantor set which can be extended to a homeomorphism of the real line onto itself.

9. For a given natural number $n \geq 2$ let $A_N = \{0, 1, \ldots, N-1\}$ and let Ω_N be the space of all infinite sequences of elements from the "alphbet" A_N. Define the distance in Ω_N as follows: for $\omega = (\omega_1, \omega_2, \ldots)$ and $\omega' = (\omega_1', \omega_2', \ldots)$:

$$\operatorname{dist}(\omega, \omega') = \sum_{n=1}^{\infty} \frac{|\omega_n - \omega_n'|}{2^n}.$$

Prove that this makes Ω_N into a Cantor space i.e. a metric space homeomorphic to the Cantor set.

10. For a given prime number p define the *p-adic norm* $\| \ \|_p$ on the field $\mathbb{Q}$ of rational numbers by

$$\|r\|_p = p^{-m}, \text{ if } r = p^m \frac{k}{l}, \text{ where } m, k, l \in \mathbb{Z} \text{ and } k, l \text{ are relatively prime with } p.$$

Prove that the distance function $d(r, r') = \|r - r'\|_p$ defines a metric on $\mathbb{Q}$.

11. The completion of $\mathbb{Q}$ in the metric defined in the previous problem is called *p-adic numbers* and the completion of $\mathbb{Z}$ is called *p-adic integers*. Prove that the space of p-adic integers is a Cantor space and the space of p-adic numbers is homeomorphic to the disjoint union of countably many Cantor spaces.
Hint: Use the fact that integers lie in the unit ball around zero.

12.Prove that the function on $[0, 1]$ defined by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q}, p \geq 0, q > 0, (p, q) = 1 \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is Riemann integrable.

13.Prove that the function on $[0, 1]$ defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is not Riemann integrable.

14.Prove that any function on $[0, 1]$ which has (finite) limits from the left and form the right an every point is Riemann integrable.
Hint: Consider the set of point where the *oscillation* of such a function is greater than $\epsilon > 0$.

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PROBLEM SET # 2

MEASURE THEORY I.

Due on Friday 9-27-96

15. Halmos, p.31, N4.

16. Halmos, p.36, N3.

17. Halmos, p.40, N.10.

18. Halmos, p.43, N4.

19. Halmos, p.52, N6.

Given an algebra $\mathfrak{A}$ of subsets of a set X with a finite measure m we will call the metric space of equivalence classes of sets in $\mathfrak{A}$ *the metric algebra of* $(\mathfrak{A}, m)$. The measure m is called *atomic* if any set which does not contain an atom has measure zero.

20. The metric algebra of $(\mathfrak{A}, m)$ is compact if and only if $\mathfrak{A}$ is a σ–algebra and m is an atomic measure.

21. If $\mathfrak{A}$ is a σ–algebra and m is an atomic measure with infinitely many atoms that the metric algebra of $(\mathfrak{A}, m)$ is a Cantor space.

A measure is called *non–atomic* if there are no atoms.

22. Give an example of a non–atomic σ–finite Borel measure on the line such that any interval has infinite measure.

23. Show that for any non–atomic σ–finite Borel measure m on the line there exists a measurable set A such that for any interval Δ with $m(\Delta) \neq 0$ both sets $\Delta \cap A$ and $\Delta \setminus A$ have non–zero measure.
Hint: First do this for Lebesgue measure.

24. Prove that Lebesgue measure is the only finite Borel measure m on the unit interval satisfying the following property: if $\Delta \subset [0, 1]$ is an interval and r a rational number such that $\Delta + \{r\} \subset [0, 1]$ then $m(\Delta + \{r\}) = m(\Delta)$.

25. Does the assertion of the previous problem hold if one replaces "finite" in the assumption to
(i) σ–finite, (ii) non–atomic σ–finite?

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PROBLEM SET # 3

MEASURABLE FUNCTIONS AND INTEGRATION.

Due on Monday 10-14-96

26. Let (X, μ) be a measure space. Introduce the following distance in the space $M(X, \mu)$ of equivalence classes mod 0 of measurable functions:

$$d(f, g) = \text{ess sup} \frac{|f-g|}{1+|f-g|}.$$

Prove that $M(X, \mu)$ becomes a complete metric space.

27. Formulate and prove the generalization of the Lebesgue Density Point Theorem for an arbitrary locally finite (but not necessarily non–atomic) Borel measure on the real line.

27A*. (OPTIONAL) Prove two versions of the Lebesgue Density Point Theorem for Lebesgue measure on the plane: (i) for squares with sides parallel to coordinate lines centered at a point; (ii) for discs, centered at a point.
Hints: For (i) use isomorphism between Lebesgue measures in dimension one and two given by mixing the binary representations of the two coordinates; deduce (ii) from (i).

28. Halmos, p.83, N4 (You should read and understand the construction in N3 first)

29.Halmos, p.90, N2

30.Halmos, p.94, N1e

31.Halmos, p.94, N5

32.Prove existence of a measurable function $f : [0, 1] \to [0, 1]$ such that for any two subintervals I, J of $[0, 1]$, $\lambda(f^{-1}(I) \cap J) > 0$

33. Let (X, μ) be a finite measure space. Introduce the following distance in the space $M(X, \mu)$ of equivalence classes mod 0 of measurable functions:

$$d_1(f, g) = \int_X \frac{|f-g|}{1+|f-g|} d\mu.$$

Prove that $M(X, \mu)$ becomes a complete metric space. and that convergence with respect to this metric is convergence in probability.

34.Halmos, p.114, N2.

35.Halmos, p.115, N5

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PROBLEM SET # 4

Part 1: L^p AND HILBERT SPACES.

Due on Wednesday 10-30-96

36.Rudin p. 72, N7.

37.Rudin p. 72, N9.

38.Rudin p. 72, N11.

39.Rudin p. 72, N14.

40.Rudin p. 94, N17.

41.Rudin p. 94, N19.

42. Let $a = (a_1, \dots) \in l^2$. Consider the following subset $C_a \subset l^2$:

$$\{x = (x_1 \dots) : |x_n| \le a_n,\ n = 1, \dots\}.$$

Prove that C_a is compact and is homeomorphic to the product of the countable number of copies of the unit interval with the product topology.

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PROBLEM SET # 4

Part 2: MORE HILBERT SPACES.

Due on Friday 11-1-96

43. Prove that in x_n is a sequence of elements of a Hilbert space which converges weakly to an element x and $\lim_{n\to\infty} \|x_n\| = \|x\|$ then x_n converges to x, i.e $\|x_n - x\| \to O$.

44. Prove that a convex weakly closed subset of a Hilbert space is closed.

45. GIve an example of a trigonometric series which converges in L_2 (and hence is a Fouries series of an L_2 function) but which diverges in a dense set of points.

46. Consider the closure of the space of trigonometric polynolials with respect to the following scalar product: for $f = \sum_{|m|\leq N} f_m \exp 2\pi i m x$ and $g = \sum_{|m|\leq N} g_m \exp 2\pi i m x$,

$$< f, g >= \sum f_m \overline{g}_m (1 + |m|^{2k+1}).$$

Prove that every element of the completion may be naturally indetified with a function with has k derivatives.

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PROBLEM SET # 5

ABSOLUTELY CONYINUOUS AND SINGULAR MEASURES.

Due on Monday 11-18-96

47.Halmos, p.135, N3

48.Halmos, p.135, N5

49.Rudin, p.133, N8

50.Rudin, p.133, N9

51. Formulate and prove proper generalizations of the Radon–Nykodim Theorem and Lebesgue Decomposition Theorem for signed measures.

52. Let μ be a singular non-atomic measure on the interval $[a, b]$ and let F_μ be it distribution function: $F_\mu(t) = \mu([a, t])$. Prove that there are uncountably many poins where F_μ has infinite derivative.
Hint: Use Vitaly covering lemma.

53. Let μ be a probability Borel measure on the circle, i.e. $\mu(S^1) = 1$, and let μ_n, $n \in \mathbb{Z}$ be its Fourier coefficients. Assume that for some sequence $n_k \to \infty$, $|\mu_{n_k}| \to 1$. Prove that the measure μ is singular.

54. Let $F : S^1 \to S^1$ be a Lebesgue measurable map such that $F(z_1 z_2) = f(z_1)f(z_2)$. Prove that $F(z) = z^n$ for some $n \in \mathbb{Z}$.

55. Consider a continuos function f on the unit interval with the following property. For any set A of Lebesgue mesure zero, $F(A)$ also has Lebesgue measure zero. Is it true that almost everywhere

$$f(x) = \int_0^x f'(t)dt \ ?$$

In other words, is f absolutely continuous?

56. Show that the conclusion of the Radon-Nykodym theorem does not hold for *not σ–finite* measurers, i.e construct an example of two measures μ and ν on a space X such that $\mu(A) = 0$ implies $\nu(A) = 0$ but there is no non–nerative function h (even allowing that it can take infinite values) such that $\mu = h\nu$.
Hint: You may construct such measures on demand but you may also consider Hausdorff mesures with different values of the exponent.

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PROBLEM SET # 6

RIESZ AND FUBINI THEOREMS.

Due on Wednesday 12-4-96

57. Let X be a Banach space with the norm $\|\cdot\|_X$ and Y be its dense linear subspace provided with a complete norm $\|\cdot\|_Y$, such that $\|x\|_Y \geq \|x\|_X$. Obviously any continuous linear functional on $(X, \|\cdot\|_X)$ determines a continuous linear functional on $(Y, \|\cdot\|_Y)$. Prove that there exists a continuous linear functional on $(Y, \|\cdot\|_Y)$ which does not extend to a continuous linear fucntional on $(X, \|\cdot\|_X)$.

58. Consider the space $C^1([0,1])$ of all continously differentiable functions on the interval $[0,1]$ with the norm $\|f\| = \sup|f| + \sup|f'|$. Find a general form of a continuous linear functional on the space $C^1([0,1])$. Give an example of a functional satisfying the assertion of the previous problem.

59. Consider the space $CL([0,1])$ of all bounded functions on the unit interval which have finite limits from the left and from the right at every point and are continuous from the left, provided with the sup norm. Find a general form of a continuous linear functional on $CL([0,1])$.

60.Let X be a compact metric space which contains infinitely many points and $C^*(X)$ be the dual space to the space $C(X)$, i.e. the space of all finite Borel signed mearures on X with the variation norm. Prove that there exists a continuous linear functional l on $C^*(X)$ which does not have a form $l(\mu) = \int_X f d\mu$ for some continuous function f. In other words, $C(X)^{**}$ is strictly greater than $C(X)$.

61. Under the assumptions of the previous problem give an example of a space X such that any $l \in C^{**}(X)$ has the form

$$l(\mu) = \int_X f d\mu$$

for some *Borel* function on X.

62. Let ν_x, $x \in [0,1]$ be a family of bounded measures on the interval $[0,1]$ continuous in the weak *-topology. Prove that there exists a unique Borel measure μ on the unit square $S = [0,1] \times [0,1]$ such that for any continuous function $f \in C(S)$,

$$(*) \qquad \int_S f d\mu = \int_0^1 \left(\int_0^1 f(x,y) d\nu_x(y) \right) dx.$$

Prove in addition that the measure μ is non-atomic.

64. Give an example of a finite Borel measure on S which has the form $(*)$ without the weak continuity assumption and such that the following extra properties are satisfied:
(i) The measures ν_x are atomic.
(i) The measure μ is positive on any non–empty open set.

64. Halmos, p.149, N5.

65. Halmos, p.150, N9.