MATH 501: REAL AND COMPLEX ANALYSIS

FALL 1996

A.Katok

MIDTERM EXAMINATION

Saturday October 12, 1996, 10am–12:15pm.

You may refer to the theorems proved in class, to standard facts from the first five chapters of Halmos and to the statements of all regular (but not optional) homework problems.

For a perfect score you should give complete solutions of two problems from Section 1 and one problem from Section 2.

SECTION 1

1.1.Consider a bounded mesurable function f on the real line which takes no more than countably many values. Prove that for almost every point x,

$$\lim_{\epsilon\to 0}\frac{\int_{x-\epsilon}^{x+\epsilon} f(x)dx}{2\epsilon} = f(x).$$

1.2.Let C be the product of countably many copies of the unit interval with the Lebesgue measure λ. Let μ be the product measure on C. Construct an isomorphism between measure spaces $([0,1],\lambda)$ and (C,μ).

1.3. Prove that a uniformly bounded sequence of functions on a finite measure space which converges in measure is a Cauchy sequence in the L_p norm for any $1 \le p < \infty$.

1.4.Let λ be Lebesgue measure on the plane and A be a measurable set. Prove that for any $\epsilon > 0$ there exists a ball B of arbitrary small radius such that

$$\frac{\lambda(A\cap B)}{\lambda(B)} > 1-\epsilon.$$

Hint: First prove this for small squares.

SECTION 2

2.1.Using Hahn–Banach Theorem prove that Lebesgue measure on the unit interval can be extended to a *finitely additive* measure defined on the algebra of *all* subsets of the interval.

2.2.Prove that for the space $C[0,1]$ of all continuos functions on the unit interval the standard embedding into the second dual is not a bijection.