ANALYSIS: MATH 501 — FIRST MIDTERM EXAM

1. *Prove that in* $\mathbb{Q}_5$ *there exists a square root of* $-1(= \ldots 44)$*, and find its last three digits.*

How many such roots are there? Fully explain your answer.

2. *Suppose that a measure* μ *is defined on a semi-ring of subsets of a given set* X*, containing* X *itself, and let* μ^* *be its outer measure. A set* $A \subset X$ *is said to be measurable in the Carathéodory sense if for any subset* $Z \subset X$

$$\mu^*(Z) = \mu^*(Z \cap A) + \mu^*(Z \setminus A).$$

Prove that A *is Lebesgue measurable if and only if it is measurable in the Carathéodory sense.*

3. *Prove that a simple function is measurable if and only if all its level sets*

$$L_c(f) = \{x \in X \mid f(x) = c\}$$

are measurable.

Is this true for arbitrary functions? Either prove or give a counter-example.

4. *Define a function* $f(x)$ *on* $[0,1]$ *as follows. If* $x = .n_1 n_2 \ldots$ *is the decimal expansion of* x*, then* $f(x) = \max_i n_i$*. Prove that* $f(x)$ *is measurable and almost everywhere constant.*

ANALYSIS: MATH 501 — SECOND MIDTERM EXAM

1. *Give a direct proof of the following statement: the integral of a nonnegative function f over a set A is equal to 0 only if $f(x) = 0$ almost everywhere on A.*

2. *Determine whether the function $\phi(f)$ has bounded variation on $[0,1]$ if f has bounded variation on $[0,1]$ and ϕ*

(1) *is continuous on the whole real line;*
(2) *has bounded variation on the whole real line.*

3. *Let $X = \mathbb{Z}_p$ be the set of p-adic integers, S the algebra of subsets of X that are simultaneously open and closed in X with respect to topology given by the metric d_p, and $\mathcal{U} = R_\sigma(S)$.*

(1) *Prove that any closed ball in X is open and hence belongs to S.*
(2) *Prove that any open set belongs to $\mathcal{U}$.*

4. *Prove the Lebesgue Dominated Convergence Theorem for the convergence in measure: Let $\{f_n\}$ be a sequence of μ-integrable functions on a set X that are bounded in modulus by a fixed non-negative μ-measurable function ϕ and that converge in measure μ on X to a function f. Then f is μ-integrable on X, and*

$$\lim_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu$$

for any μ-measurable set A.