ANALYSIS: MATH 501 — FINAL EXAM

INSTRUCTIONS

Choose 3 problems out of problems 1-6 and 1 problem out of problems 7-8

1. *Let $f(x)$ be a function differentiable everywhere on $[0,1]$. Prove that $f'(x)$ is measurable.*

2. *Let $0 < \alpha \leq \beta < \infty$. For which p does the function*

$$f(x) = \frac{1}{x^\alpha + x^\beta}$$

belong to $L_p(\mathbb{R}_+, \mu)$, where $\mathbb{R}_+$ denotes the positive half-line and μ the Lebesgue measure?

3. *Find the domain of convergence in the field $\mathbb{Q}_p$ of the series*

$$\sum_{k=1}^{\infty} \frac{x^k(-1)^{k-1}}{k}.$$

4. *A monotone function $f(x)$ on $[0,1]$ is called singular if $f' = 0$ a.e.*

(1) *Show that any monotone increasing function is the sum of an absolutely continuous function and a singular function.*
(2) *Give an example of a function which is monotone and continuous on $[0,1]$ but not absolutely continuous on any subinterval of $[0,1]$.*

5. *Let*

$$f(x,y) = \begin{cases} 2^{2n}, & \text{for } \frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}}, \frac{1}{2^n} \leq y \leq \frac{1}{2^{n-1}}, \\ -2^{2n+1}, & \text{for } \frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n}, \frac{1}{2^n} \leq y \leq \frac{1}{2^{n-1}}, \\ 0, & \text{otherwise.} \end{cases}$$

Is $f(x,y)$ integrable on $[0,1] \times [0,1]$?

6. *Give an example of a sequence $\{f_n\}$ of Lebesgue integrable non-negative functions on $[0,1]$ such that*

(1) $f_n \to f$ *a.e. on* $[0,1]$,
(2) $\underline{\lim} \int_0^1 f_n(x)dx < \infty$,

but $\int_0^1 f(x)dx < \underline{\lim} \int_0^1 f_n(x)dx$.

7. *If $f(z)$ is holomorphic in a domain Ω and is not constant, show that Ref attains neither a maximum nor a minimum in Ω.*

8. *Let $\Omega = \{0 < |z| < 2\}, f \in H(\Omega)$. Is it true that if*

$$\int_{|z|=1} f(z)dz = 0$$

then f has a removable singularity at $z = 0$?