Algebra Qualifying Exam
May 1999

Do 6 of the following problems.

1. Let G be a finite group of order 992. Prove that G is not simple.

2. List all isomorphism classes of groups of order 8 and show that your list is complete.

3. Consider $\mathbf{S}_4 \subset \mathbf{S}_5$ where $\mathbf{S}_4$ is the group of permutations of $\{1,2,3,4\}$ and $\mathbf{S}_5$ is the group of permutations of $\{1,2,3,4,5\}$. The group $\mathbf{S}_4 \times \mathbf{S}_4$ acts on $\mathbf{S}_5$ via $(\sigma_1, \sigma_2)(\tau) = \sigma_1 \tau \sigma_2^{-1}$. Exhibit $\mathbf{S}_5$ as a disjoint union of $\mathbf{S}_4 \times S_4$-orbits and give the number of elements in each orbit.

4. Let $V \subset \mathbf{R}[X,Y]$ be the space of polynomials in X, Y of total degree $\leq n$,

$$A : V \to V\,, \qquad f \mapsto f + \partial f/\partial X + \partial f/\partial Y\,.$$

   Find the Jordan form of A.

5. Let F be a finite field with q elements, and let

$$S_1 = \{T + a \mid a \in F\}\,,$$
$$S_2 = \{T^2 + aT + b \mid a, b \in F\}$$
$$S_3 = \{T^3 + aT^2 + bT + c \mid a, b, c, \in F\}\,.$$

   Determine the number of irreducible polynomials in each of S_1, S_2, S_3.

6. How many solutions (x, y) does the equation

$$xy = 0$$

   have in the following rings: a) $\mathbf{Z}_{49}$; b) $\mathbf{Z}_{82}$?

7. Determine the group of units of the ring $\mathbf{Q}[T]/(T^2+1)^2$.

8. Let $R = \{a + b\sqrt{-5} \quad a, b \in \mathbf{Z}\}$. Show that R is not a prinicpal ideal domain

9. Given an example of Galois extension $E/\mathbf{Q}$ of degree 5.

10. Let R be a commutative ring, and let J be the set of idempotents of R. Define commutative binary operations $\oplus$ and $*$ on J by

$$e \oplus f = e + f - ef\,, \qquad e * f = e\,f.$$

   Show that $(J, \oplus, *)$ is a commutative ring, i.e. verify that:
   (a) J is closed under $\oplus$ and $*$;
   (b) $\oplus$ and $*$ are associative;
   (c) $e * (f \oplus g) = e * f \oplus e * g$.