Qualifying Exam in Algebra

May 6,1996

Do six of the following ten problems. In this exam, $\mathbf{Z}$ denotes the integers, $\mathbf{Q}$ the rationals, and $\mathbf{C}$ the complex numbers.

1. Let G be a group of order 60 containing a normal subgroup of order 5. Show that G contains an element of order 15.

2. Let G be a group and $H \neq G$ be a subgroup which contains every subgroup $K \neq G$ of G. What can you say about G?

3. Let G be a finite simple group and p a prime number. Suppose that G has exactly $p+1$ Sylow p-subgroups. Show that p^2 does not divide the order of G.

4. Which of the following 4 matrices are similar over $\mathbf{Z}$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}?$$

Which of them are similar over $\mathbf{Z}/2\mathbf{Z}$?

5. Let A be a subring of the complex numbers consisting of all numbers $a+bi$ with integer a, b. Compute the order of the multiplicative group of the ring $A/13A$.

6. Let A be the ring of all 2×2 real matrices. Prove that the center of A consists of all matrices $(xy - yx)^2$ with x, y in A.

7. Let R be an associative ring with 1, M a right R-module, $F : M \to R$ a homomorphism of R-modules with $f(M) = R$. Prove that there is a decomposition $M = K \bigoplus L$ with $f(K) = 0$ and $f|_L : L \to R$ is an isomorphism.

8. Find the Galois group of the polynomial $3x^3 - 9x^2 + 9x - 5$ over $\mathbf{Q}$.

9. Let E be a splitting field of $x^8 - 1$ over a field F of 4 elements. Find card(E).

10. Let F be a field, and $f(x) \in F[x]$ a nonzero monic polynomial. Suppose that the zeros of $f(x)$ in a splitting field E of $f(x)$ over F are all distinct and that the set of zeros is closed under multiplication. Prove that either $f(x) = x^n - 1$ or $f(x) = x^n - x$ for some natural number n.