**Qualifying Exam in Algebra**

May 9,1994

Do six of the following ten problems

1. Show that whenever $G$ is a group and $g \in G$, then

$$\{x \in G | x^{-1}gx \in \{g, g^{-1}\}\}$$

is a subgroup of G.

2. Prove that every group of order 35 must be cyclic.

3. If $G$ is an abelian group generated by two elements $a$ and $b$ satisfying $a^{20} = 1$, $b^{30} = 1$, and $a^5 = b^3$, what is the largest possible order of $G$?

4. Let $R$ be a ring containing the rational numbers as a subring (with 1). Suppose $r \in R$ satisfies $r^n = 0$ for some positive integer $n$. Prove that

$$s = 1 + \frac{r}{1!} + \frac{r}{2!} + \ldots + \frac{r^{n-1}}{(n-1)!}$$

is invertible in $R$.

5. Give an example of a commutative ring (with 1) having a prime ideal that is nonzero and not maximal. Justify your answer.

6. Let $V$ be a finite-dimensional vector space over a field $F$, and let $T : V \to V$ be a linear transformation. Suppose its characteristic polynomial is reducible over $F$. Prove that there is a nontrivial invariant subspace (that is, a subspace $0 \neq W \neq V$ with $T(W) \subset W$).

7. Let $F$ be a field of characteristic $p > 0$. Show that the multiplicative group of $F$ contains no elements of multiplicative order $p$.

8. Let $m$ and $n$ be positive integers. Let p be a prime, $F$ a field with $p^n$ elements. Define $\phi : F \to F$ by

$$\phi(b) = b^{p^m}.$$

Determine the number of elements fixed by $\phi$.

9. Let $F$ be a field, $K$ be a separable extension of degree $n$. Find some integer $b$ (depending on $n$) for which you can prove that $K$ has at most $b$ subfields containing $F$.

10. Find an irreducible monic polynomial of degree 3 over the field $F = Z/5Z$. Justify your answer.