Qualifying Exam in Algebra

May 10,1993

Do six of the following ten problems. In this exam, $\mathbf{Z}$ denotes the integers, $\mathbf{Q}$ the rationals, and $\mathbf{C}$ the complex numbers.

1. Let G and H be non-abelian simple groups. Prove that the only normal ubgroups of $G \times H$ are $\{1\}, G, H$ and $G \times H$.

2.Let G be a group and H, K subgroups of G. For $y \in G$, let $HyK = \{hyk | h \in H, k \in K\}$. If G is finite, show that $|HyK| = |H|[K : (y^{-1}Hy) \cup K]$.

3. Let $A = G \times G \times G$, where $G = \mathbf{Z}/p\mathbf{Z}$ and p is a prime number. Find $|AutA|$.

4. Classify the groups of order $7^2 \cdot 11^2$.

5. Let R be the following subgroup of $\mathbf{Q}$: $R = \{a/b | a, b \in \mathbf{Z}$ and $b \neq 0$ is odd $\}$. Prove that every ascending chain of ideals in R is finite (i.e., R is Noetherian), but R has infinite descending chaim of ideals (i.e., R is not Artinian).

6. Let $R \neq 0$ be a ring with $1 \in R$. Prove that R is a division ring if and only if the only one-sided ideals of R are 0 and R. `Note:` R is not assumed to be commutative; in order to show that $x \neq 0$ is invertible you must find an element $y \in R$ such that $xy = 1 = yx$.

7. Find the number of *distinct* roots of $x^4 + 2x^3 + 4x = 3$ in $\mathbf{C}$. Justify your answer.

8. Let C_3 be the cyclic group of order 3. Prove that there is a normal and separable field extension $E/\mathbf{Q}$ such that $Gal(E/\mathbf{Q}) \approx C_3 \times C_3 \times C_3$.

9. Let $F = \mathbf{Z}/13\mathbf{Z}$ and $K = F(x)$, where x is transcendental over F. Let $\sigma : K \to K$ be the F-automorphism of K induced by $\sigma(x) = x + 1$. Prove that the invariant subfield of σ is Inv $\sigma = F(y)$, where $y = x^{13} - x$.

10. Let $K = \mathbf{C}(x)$, x transcendental over $\mathbf{C}$, and let $\alpha \in \mathbf{C}$ satisfy $\alpha^3 = 1, \alpha \neq 1$. Let τ, η be the automorphisms of K over $\mathbf{C}$ such that $\tau(x) = \alpha x$ and $\eta(x) = x^{-1}$. Show that the group G of automorphisms generated by τ and η has order 6 and Inv $G = \mathbf{C}(x^3 + x^{-3})$.