Qualifying Exam in Algebra, May 18, 2007

Do six of the following ten problems. Explain your answers.

$\mathbb{Z}$ = ring of integers. $\mathbb{Q}$ = rational field. $\mathbb{R}$ = real field.

1. Let $G \times G$ be the direct product of a group G with itself. Suppose that $\varphi : G \times G \to G$ is a group homomorphism, and there is an element $u \in G$ such that $\varphi(u, x) = x = \varphi(x, u)$ for all $x \in G$. Show that G is abelian and φ is the group multiplication.

2. Let S_{10} be the symmetric group of permutations of $\{1, 2, \dots, 10\}$. What is the largest integer k such that S_{10} has an element of order k?

3. Let p, q be distinct primes. Prove that no group of order p^2q is simple.

4. Let G be a group of order $2007 = 3 \cdot 3 \cdot 223$ (223 is a prime). Suppose that G acts on a set S of size 425. Prove that at least one element in S is fixed by G.

5. Let V be a finite-dimensional vector space over $\mathbb{R}$, and let $T : V \to V$ be a linear transformation such that

$$T^4 - 1 = T^3 + 3T^2 + T + 3I = 0.$$

Prove that V is a direct sum of 2-dimensional T-invariant subspaces.

6. Suppose an abelian group A is generated by elements a, b, c, d subject to defining relations

$$\begin{aligned} 2a + b - 3c + d &= 0, \\ 4a + 6c - 2d &= 0, \\ -b + c + d &= 0. \end{aligned}$$

Find a product of cyclic groups isomorphic to A.

7. Let R be ring. Suppose that there are maximal left ideals $M_1, \dots, M_n$ in R such that $M_1 \cap \cdots \cap M_n = 0$. Prove that R is semisimple as a left module (i.e., a direct sum of simple left R-modules). You may use general properties of semisimple modules, provided that you state these properties explicitly.

8. Let F be the field with 2 elements, and let $L = F(t)$, the field of rational functions in a variable t. Let σ be the automorphism of L sending t to $t + 1$, and let K be the subfield of L fixed by σ.

(a) What is $[L : K]$, the dimension of L over K?

(b) What is the minimal polynomial of t over K?

9. Let $f(x)$ be an irreducible polynomial in $K[x]$, where K is a field of characteristic zero. Let α be a root of $f(x)$ in splitting field L of $f(x)$. Show that for all nonzero $\beta \in K$, $\alpha + \beta$ is not a root of $f(x)$.

10. Let K be a splitting field of $x^{33} - 1$ over $\mathbb{Q}$. Determine the number of subfields of K (including $\mathbb{Q}$ and K).