Algebra Qualifying Exam
May 9, 2004

Do six of the following ten problems. In this exam, C_n will denote the cyclic group of order n, R will denote a commutative ring with unit, $\mathbb{Q}$ will denote the rational numbers.

1) Let D be a unique factorization domain and I a non-zero principal ideal: $I = (c) \neq 0$. Prove that the quotient ring D/I contains a non-zero nilpotent element if and only if there is a prime element $p \in D$ with $p^2 \mid c$.

Solution: $\Leftarrow$: $p^2 \mid c$, so $c = p^2 a$, some $a \in D$. Put $b := pa$, which is not in I, while $b^2 = p^2a^2 = ac \in I = (p^2 a)$. Hence in D/I, $\bar{b} \neq 0$, while $\bar{b}^2 = 0$.
$\Rightarrow$: Assume that $\bar{b}^n = 0$ in D/I, but $\bar{b} \neq 0$. That is, $c \mid b^n, n \geq 2$, $c \not| b$. Let p be a prime divisor of c. Then $p \mid b^n \Rightarrow p \mid b \Rightarrow p^2 =\mid b^2 \mid b^n$, where the first implication follows from the fact that p is prime.

2) Let K be an *algebraically closed* field of finite transcendence degree over the subfield F. Show that any homomorphism $f\colon K \to K$ which restricts to the identity map on F is bijective on K. Solution: Since K is a field, $\operatorname{Ker} f = \{0\}$, i.e. f is injective. Let $t_1, \ldots, t_n$ be a transcendence basis for K/F. Since f is injective, the images $u_1 = f(t_1), \ldots, u_n = f(t_n)$ must be algebraically independent over F and therefore a transcendence basis for the algebraically closed field $f(K)$. (This latter assertion uses some fact such as that an algebraic extension field containing roots of all polynomials over the base field is itself algebraically closed.)
Now since n is finite and the transcendence degree of an extension is well-defined, K is algebraic over $F(u_1, \ldots, u_n)$. Hence K lies in the algebraic closure, $f(K)$. But we are given that f maps K to itself, i.e. $f(K) \subset K$. Therefore $K = f(K)$.

3) Determine the splitting field for $f(x) = (x^2 - 7)(x^2 + x + 2)$ over the rational field $\mathbb{Q}$. Describe explicitly the elements in its Galois group, and list the subgroups and the corresponding intermediate fields.

Solution: The roots generate the same field as $\pm\sqrt{7}, \pm\sqrt{-7}$. So the splitting field is $K = \mathbb{Q}(\sqrt{7}, \sqrt{-7}) = \mathbb{Q}(\sqrt{7}, i)$, a degree 4 extension of $\mathbb{Q}$. The four elements, together with corresponding fixed fields are:

$$\begin{array}{ccc} 1 & \leftrightarrow & \mathbb{Q}(\sqrt{7}, i) \\ \{1, \text{complex conjugation}\} & \leftrightarrow & \mathbb{Q}(\sqrt{7}) \\ \{1, \sqrt{7} \mapsto -\sqrt{7}\} & \leftrightarrow & \mathbb{Q}(i) \\ \{1, (i, \sqrt{7}) \mapsto (-i, -\sqrt{7})\} & \leftrightarrow & \mathbb{Q}(i\sqrt{7}) \end{array}$$

4) Find the total number of subgroups of $C_3 \times C_3 \times C_3$. Justify your answer.

Cheap trick: Consider as $\mathbb{F}_3$ vector spaces. Then the subgroups are the $\mathbb{F}_3$-subspaces:

dimension	number	by counting
0	1	
1	$(3^3-1)/2$	either non-zero generator
2	$(3^3-1)/2$	either non-zero perpendicular element
3	1	

,

where the third line comes from noting that the subspace satisfies *one* linear equation over $\mathbb{F}_3$. Total number: 28 5) Determine all finite fields K for which the mapping $f\colon K \to K$

defined by $f(x) = x^3$ is a field automorphism.

Solution: Note that $1+1 = f(1)+f(1) = f(2) = 2^3$. Hence $6 = 0$, and the characteristic of K must be either 2 or 3. Notice also that, since f is surjective, every element in K has a cube root in K.
For arbitrary α, $1+\alpha^3 = f(1)+f(\alpha) = f(1+\alpha) = (1+\alpha)^3$, so $3\alpha(1+\alpha) = 0$.
If $\operatorname{char} K = 2$, then $\alpha = 0, 1$, i.e. $K = \mathbb{F}_2$.
If $\operatorname{char} K = 3$, then this is the 3-power Frobenius map, the generator of the Galois group of the finite field K over $\mathbb{F}_3$, and K is an arbitrary finite extension of $\mathbb{F}_3$.

6) Let A be a complex $n \times n$ matrix of rank 1.
a) What are the possible Jordan canonical forms for A? Justify your answer.
b) For each form found, compute the characteristic and minimal polynomials.

Solution: If A has rank one, then so does its Jordan canonical form. Thus any non-zero Jordan block J must be either $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} \alpha \end{pmatrix}$ for non-zero α.

This is true because any $i \times i$ Jordan block $\begin{pmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ & \dots & \dots & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$, will have rank greater than 1 if either $i > 2$ or else $i = 2$ and $\lambda \neq 0$.
Thus the Jordan canonical form is $\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$. Let us consider the cases according to J:
$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ gives characteristic polynomial $P_A(t) = t^n$ and minimal polynomial t^2.
$J = \begin{pmatrix} \alpha \end{pmatrix}$, for non-zero α, has characteristic polynomial $P_A(t) = t^{n-1}(t-\alpha)$ and minimal polynomial $t(t-\alpha)$ when $n > 1$.
$\begin{pmatrix} \alpha \end{pmatrix}$ for non-zero α has minimal and characteristic polynomials $P_A(t) = t - \alpha$.

7) Show that a group G of order 36 must have a normal subgroup of order 3 or 9 .

Solution: Let n_3 denote the number of 3-Sylow subgroups of G. Then $n_3 \equiv 1 \mod 3$ and $n_3 \mid 36$. The positive divisors of 36 satisfying the congruence are 1 and 4.
In the first case, the unique 3-Sylow subgroup is normal. In other words, we have a normal subgroup of order 9.

In the second case, letting G act on the set of 3-Sylow subgroups by conjugation gives a homomorphism $G \to S_4$, which acts transitively on the 4 3-Sylow subgroups. This last fact gives that the normalizer of such a 3-Sylow subgroup has index 4, i.e. is the subgroup itself. Therefore the image must have order dividing both $4! = 24$ and 36, i.e. dividing their GCD, 12.

However, in order to act transitively on the 4 3-Sylow subgroups, the image must have a subgroup of index 4. The possible orders are $4, 12$. In the first case, the kernel of the map to S_4 is a normal subgroup of order 9; in the last, of order 3.

Remark: A_4 has 4 3-Sylow subgroups. So the latter case can occur: $G = C_3 \times A_4$.

8) Let M be an R-module. Show that the R-subalgebra $S = R \oplus \bigwedge^2 M \oplus \bigwedge^4 M \oplus \bigwedge^6 M \oplus \ldots$ lies in the center (the subalgebra commuting with all elements) of the exterior algebra $\bigwedge M = R \oplus \bigwedge M \oplus \bigwedge^2 M \oplus \bigwedge^3 M \oplus \ \ldots$.

Solution: Note that $u \in S$ will lie in the center iff it commutes with all the homogeneous elements of $\bigwedge M$ iff each homogeneous component of u does so. However, in general if $u \in \bigwedge^p M$ and $v \in \bigwedge^q M$, then a theorem from class tells us that $u \wedge v = (-1)^{pq} v \wedge u$. So whenever $u \in \bigwedge^p M$ and p is even, $v \wedge u = u \wedge v$ for every $v \in \bigwedge^q M$ regardless of q and thus for every $v \in \bigwedge M$ One can also do this directly, essentially reproving the theorem cited. By multi-linearity of the exterior (wedge) product, it is enough to work with decomposable elements.

9) Prove that two 3×3 nilpotent matrices with entries from the same field K are similar if and only if they have the same minimal polynomial.

Solution: Similarity classes are determined by the invariant factors, call them $a_1(t), \ldots, a_s(t)$. The minimal polynomial is $a_s(t)$ and their product is the characteristic polynomial: $P(t) = a_1(t) \ldots a_s(t)$.

Nilpotence tells us that the characteristic polynomial here is $P(t) = t^3$. So the possibilities for the invariant factors are:

s	$a_1(t), \ldots, a_s(t)$
1	t^3
2	t, t^2
3	t, t, t

We recognize that the cases are indeed distinguished by $a_s(t)$, the minimal polynomial.

10) Let the abelian group A be generated by the elements a, b, c satisfying the defining relations:

$$
\begin{aligned}
2a - 4b + 10c &= 0 \\
6b - 6c &= 0 \\
12c &= 0
\end{aligned}
$$

Give explicitly in terms of the generators a, b, c two elements of order 3 which are not multiples of each other in A . Justify your answer.

Solution: Carrying out row reduction (no column operations are necessary in this example) or otherwise, we find the Smith Normal Form of the matrix of relations:

$$\begin{pmatrix} 2 & 0 & 0 \\ -4 & 6 & 0 \\ 10 & -6 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ -4 & 6 & 0 \\ 10 & -6 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -1 & 1 \end{pmatrix}$$

¿From this, we see that $A = \langle a' \rangle \times \langle b' \rangle \times \langle c' \rangle \simeq C_2 \times C_6 \times C_{12}$, and that the elements of order 3 are the non-zero elements of $\langle 2b' \rangle \times \langle 4c' \rangle$, where $b' = b - c$, $c' = c$. Two elements of order 3 are then $2b' = 2b - 2c$ and $4c' = 4c$. From the direct sum decomposition, we see that these elements are not multiples of each other in the group.

The elements can be produced much more easily bare-hands: From the defining relations, c has order 12 and $b - c$ has order 6. Therefore $4c$ and $2b - 2c$ have order 3. However an argument must be given that neither element is a multiple of each other.