Qualifying Exam in Algebra - May 6, 2002

Do six of the following problems. If you do more than six, indicate which six you wish to be graded. Justify your answers.

1. Prove that a group G of order 1700 cannot be simple.

2. Find the largest integer which is the order of an element of S_9, the symmetric group on 9 letters.

3. Let G be a group of order p^a, where p is a prime and $a \geq 1$. Show that $Z(G) \neq \{1\}$, where $Z(G)$ is the center of G.

4. Let H and K be normal subgroups of a group G such that $H \cap K = \{1\}$. Prove that every element of H commutes with every element of K.

5. Let $R = \mathbb{C}[x, y, z]$ be a polynomial ring in three variables over the complex numbers. Show that $x^3 + y^3 + z^3$ is not equal to the sum of two cubes of elements of R. (Hint: First show that $x^3 + y^3 + z^3$ is irreducible.)

6. Let R be a commutative ring with unit, and let I be the 2×2 identity matrix. Show that if $A, B \in M_2(R)$ and $AB = I$, then $BA = I$. (Hint: det(AB) = det(A)det(B).)

7. Let A be an $n \times n$ matrix of rank one over the complex numbers. List the possible Jordan canonical forms for A, and in each case give the characteristic and minimal polynomials of A.

8. Decide whether $x^2 + x + 1$ is irreducible over F_p, the field with p elements, for all primes $p < 100$.

9. Let E be a splitting field for $x^3 - 7$ over $\mathbb{Q}$, the field of rational numbers. List all subfields of E.

10. Let F be a field of characteristic $p > 0$. Suppose that $F \subseteq F[\alpha]$, where $\alpha^p \in F$. Show that $[F[\alpha] : F]$, the dimension of $F[\alpha]$ over F, is equal to 1 or p.