Qualifying Exam in Algebra

August, 1999

Outline of Solutions

1. Let $p < q < r$ be primes, and let G be a group of order pqr. Prove that G is solvable.

Let P, Q, R be, respectively, the p,q,r-Sylow subgroups of G. If H is a t-Sylow subgroup of G, then, by Sylow's theorem, the number of t-Sylow subgroups is the index of the normalizer of H in G, and is congruent to 1 mod t.

The index of the normalizer of R in G is $1, p, q$, or pq, and p and q are not congruent to 1 mod r. Hence the number of r-Sylow subgroups is either 1 or pq. Similarly, the number of q-Sylow subgroups is 1, r, or pr, and the number of p-Sylow subgroups is 1, q, r, or qr. Since all Sylow subgroups are cyclic, the intersection of distinct Sylow subgroups is the trivial group.

We claim that at least one of P, Q, R is normal in G. If not, then G contains $pq(r-1)$ elements of order r, at least $r(q-1)$ elements of order q, and at least $q(p-1)$ elements of order p. Since

$$pq(r-1) + r(q-1) + q(p-1) = pqr + qr - q - r > pqr,$$

this is a contradiction, proving the claim. Passing to G/N, where N is whichever of P, Q, R is normal, and letting $|G/N| = st$, where $s < t$ are primes, it follows at once from Sylow's theorem that the t-Sylow subgroup of G/N is normal, proving that G is solvable.

2. Determine (up to isomorphism) all groups G which satisfy: There is a proper subgroup H of G which contains all proper subgroups of G.

Let g be an element of $G - H$. Then the group generated by g is not contained in H, so it must be all of G. Thus G is cyclic, and has a unique maximal subgroup. An easy analysis shows that G must be cyclic of order p^n for some prime p.

3. Let V be a finite-dimensional vector space over a field of characteristic $p > 0$. Let $T : V \to V$ be a linear transformation with $T^{p^n} = Id$ for some n. Prove that there is a nonzero vector v in V with $Tv = v$.

Since the characteristic is $p > 0$, $(T - Id)^{p^n} = T^{p^n} - Id = 0$. Letting j be the least integer such that $(T - Id)^j = 0$, we have that $(T - Id)(v) = 0$, where v is a nonzero element of $(T - Id)^{j-1}(V)$. (By convention, $(T - Id)^0(V) = V$.)

4. Let R be a commutative ring with unit and suppose that N is an ideal of R such that $N^2 = 0$. Let $\varphi : R \to R/N$ be the canonical map, and suppose that $e \,\epsilon\, R/N$ satisfies $e^2 = e$. Show that there exists $d \epsilon R$ such that $\varphi(d) = e$ and $d^2 = d$. (Hint: $(2e-1)^2 = 1$.)

Let c be any preimage of e. Then $c^2 - c \,\epsilon\, N$. We want to let $d = c - y$, where $y \,\epsilon\, N$ and

$$0 = d^2 - d = (c-y)^2 - (c-y) = c^2 - c - (2c-1)y.$$

Noting that $(2e-1)^2 = 1$ and $(2c-1)^2 \equiv 1 \bmod N$, we see that $y = (2c-1)(c^2-c)$, $d = c - y = -2c^3 + 3c^2$.

5. A and B agree to divide n dollars, where n is the square of a positive integer. They do so as follows: A takes ten dollars, then B takes ten dollars, and they continue alternatively to take ten dollars until it is B's turn, but there are less than ten dollars left. Then B takes the remaining money and A gives him x dollars, at which point the money has been evenly divided. Find x and justify your answer.

The fact that B has less than ten dollars to take at the final step implies that $n \equiv m \bmod 20$, where $10 \leq m \leq 19$. Modulo 20, the squares are $0, 1, 4, 5, 9$ and 16. Thus $n \equiv 16 \bmod 20$, so there are six dollars remaining for B, and A must give him two dollars to divide the money evenly.

6. Let F be a field of characteristic 3. Inside the field of rational functions $F(x)$, let E be the subfield of those f with $f(x) = f(2x+1)$. Show that $F(x)$ is a Galois extension of E, and determine its Galois group.

Let $\varphi : F(x) \to F(x)$ be the automorphism defined by $\varphi(f(x)) = f(2x+1)$. Since $2(2x+1)+1 = x$, φ has order 2 as an automorphism of $F(x)$. Now E is the fixed field of $F(x)$ under the action of φ, so $F(x)$ is Galois over E, and the Galois group is cyclic of order 2.

7. Let E be a splitting field of $x^{88} - 1$ over $GF(9)$, the field with 9 elements. Determine the cardinality of E and make a diagram showing all subfields of E and the inclusions between them.

Since the multiplicative group of the finite field $GF(3^n)$ is cyclic of order $3^n - 1$, we first find the smallest n such that $88 | 3^n - 1$, and this turns out to be $n = 10$. Since $GF(3^{10})$ contains $GF(9)$, the required field is $GF(3^{10})$. Its subfields are $GF(3), GF(3^2), GF(3^5)$, and $GF(3^{10})$, and the inclusions between them are obvious.

8. Let A_5 be the alternating group on five letters. Show that $A_5 \times A_5$ can be generated by two elements and exhibit a pair of elements which generate it.

First, choose a pair of generators for A_5 whose orders are relatively prime, say $g = (12)(34)$ and $h = (135)$. Then let $x = (g,h), y = (h,g)$, elements of $A_5 \times A_5$. Now $x^3 = (g,1)$, $x^4 = (1,h)$, $y^3 = (1,g)$, $y^4 = (h,1)$, so x and y generate $A_5 \times A_5$.

9. Determine the structure of the abelian group given by three generators x, y, z and the relations

$$7x + 2y + 3z = 0, \;\; 21x + 8y + 9z = 0, \;\; 5x - 4y + 3z = 0.$$

The relations are equivalent to

$$x + 3z = 0, 2y = 0, 6z = 0,$$

so the group is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_6$.

10. Let G be a group and let G' be the commutator subgroup of G. Suppose that G' is abelian and G/G' is a cyclic group. Show that there is some $g \,\epsilon\, G$ such that no proper normal subgroup of G contains g.

Let N be the normal subgroup of G generated by g. By, hypothesis, $G = NG'$. Let ax, by be arbitrary elements of G, where $a, b \,\epsilon\, N, x, y \,\epsilon\, G'$. Then

$$(ax)(by)(ax)^{-1}(by)^{-1} \equiv xyx^{-1}y^{-1} = 1 \bmod N,$$

so $G' \subseteq N$, and $G = NG' = N$.