Qualifying Exam in Algebra

August 20,1996 9 am - 12 noon

- Your grade will be strong pass, pass, conditional pass, or fail.
- You work alone, without receiving help or giving help to other students.
- You have 3 hours to work on 10 problems. Correct solutions of 4 problems could (and 5 problems should) be sufficient for passing. Credit for partial solutions may be given.
- You use only the paper provided and return all paper. Books, notes, and electric or electronic gadgets are not allowed except as approved by the proctor.
- Talking or other distracting activities are not allowed, except that one can ask the proctor questions in writing or in a quiet voice.
- Write down as many details of solutions as time allows and refer to nontrivial results you use.
- Good luck!

1. Let G be a finite simple group of 2×2 complex invertible matrices under matrix multiplication. Assume that the order of G is even. Show that the order of G is two. *Hint:* Consider elements of order 2 in G.

2. Two commutative groups G and G' are given by the presentations

$$\begin{pmatrix} 7 & 4 & 1 \\ 8 & 4 & 0 \\ -6 & -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 9 & 8 & 1 \\ 3 & 2 & 0 \\ -8 & -8 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

   respectively (three generators and three defining relations for each). Are they isomorphic?

3. Let H be a normal subgroup of a finite group G, $n = \mathrm{card}(H)$, $m = [G : H]$. Assume that m and n are coprime. Show that G has only one subgroup H of order n.

4. Let G be a commutative group such that $\mathrm{Aut}(G) \cong \mathbf{Z}/n\mathbf{Z}$ with $n \geq 2$. Prove that n is even.

5. Describe all two sided ideals in the ring $\mathrm{M}_2\mathbf{Z}$ of 2×2 integral matrices.

6. Consider the subring $A = \mathbf{Z} + \mathbf{Z}\sqrt{13}$ of $\mathbf{R}$. Compute the order of the multiplicative group of the ring $A/13A$.

7. Compute the degree $[\mathbf{C}(x, y) : \mathbf{C}(x^2 - y^3, y^2)]$.

8. Let G be a simple group of order 168. How many Sylov 7-subgroups and Sylov 3-subgroups are there?

9. Compute the Galois group $\mathrm{Gal}((x^2 - 1)(x^{15} - 1))$ over the field $\mathbf{Z}/3\mathbf{Z}$.

10. Is the polynomial $xy^5 - yx^2 + 1 \in \mathbf{C}[x, y]$ irreducible?