

Qualifying Exam in Algebra

August 17, 1994

Do six of the following ten problems

1. Let G be a finite a group of order greater than 1, and suppose each element of G is conjugate to its inverse. Prove that the order of G is even.
2. Prove that every group of order 164 is solvable.
3. Let G be a permutation group on n elements. Let p be a prime, $p > 2$, and suppose $n < 3p$. Prove that the Sylow p -subgroups of G are abelian.
4. Let G be an *abelian* group, generated by elements a, b, c satisfying the relations $a^2 = b^3$, $b^2 = ac$, and $a^5 = bc$. Prove that G is a finite cyclic group.
5. Over a field F , define a subring R of the 2×2 matrices by

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in F \right\}.$$

Prove that R has an ideal J such that R/J is isomorphic to the direct product of two copies of F .

6. Let A and B be two $n \times n$ matrices over a field. Suppose $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ is similar to $\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$. Prove that A is similar to B .

7. Let R be a commutative ring and i an element of R with $i^2 = i$. The R -module homomorphisms from iR to R form an R -module under function addition and scaling. Prove that this module is actually isomorphic to iR itself.

8. Find an explicit formula that expresses the determinant of every 3×3 matrix A over the rationals in terms of the traces of the powers of A . (Hint: Eigenvalues.)

9. Let F be a field with 11 elements, and let K be the splitting field of $X^3 - 1$ over F . How many roots does $(X^2 - 2)(X^3 - 2)$ have in K ?

10. Let K/F be a finite-dimensional Galois field extension with group G . Suppose b is an element of K such that $\{\sigma(b) \mid \sigma \in G\}$ is a basis of K over F . Let E be any intermediate field, and let $d = \text{Tr}_{K/E}(b)$. Show that $E = F(d)$.