Qualifying Exam in Algebra

August 19,1992

Do six of the following ten problems. Note that $\mathbf{Z}$ denotes the integers, $\mathbf{Q}$ the rationals, $\mathbf{R}$ the reals, and $\mathbf{C}$ the complex numbers

1. Show that every group of order 333 is solvable.

2.Let G be a group containing a subgroup K of index 7. Suppose K containing no subgroup which is normal in G, other than the identity subgroup. Prove that G contains no element of order 9.

3. Let G be a group of order $2^n, n \geq 2$ an integer. Suppose H and K are groups of order 2^{n-1} which are homeomorphic images of G. Prove there is a group L of order 2^{n-2} which is a homemorphic image of both H and K.

4. Let R be the subring of $\mathbf{Q}$ consisting of all rational numbers which can be written with odd denominators.
a) Prove that 2 is an irreducible element of R.
b) Prove that if p is irreducible in R then there exists an invertible element $u \in R$ such that $2 = up$.

5. Let $D = \mathbf{Q}[\mathbf{x}]$ be the polynominal ring in one variable over the field $\mathbf{Q}$ of rational numbers. Let K be the submodule of $D^{(3)} = \{(f_1, f_2, f_3) | f_i \in D\}$ generated by $(2x-1, x, x^2+3)$ and (x, x, x^2). Find polynomials $g_1, g_2, ..., g_r$ such that

$$D^{(3)}/K \approx D/(g_1) \oplus D/(g_2) ... \oplus D/(g_r).$$

6. Let $g = (-1+\sqrt{5})/2$ (the golden ratio) and let $R = \mathbf{Z}[\mathbf{g}]$. Let $f : R \rightarrow R/2R$ be the canonical homomorphism. Show that $f(g)$ is invertible and has multiplicative order 3.

7. Let F be a finite field with 2^n elemnts. Suppose F contains an element of multiplicative order 9. Show that 6 divides n.

8. Let F be a field and $f(x)$ an irreducible polynomial over F. Let $f'(x)$ be the (formal) derivative of f(x). If $f(x)$ and $f'(x)$ are **not** relatively prime in $F[x]$, show that the characteristic p of F is not 0 and that p divides the degree of $f(x)$.

9. Let $\zeta \in \mathbf{C}$ be a primitive 5th root of unity. By explicit calculation show that $\sqrt{5} \in \mathbf{Q}(\zeta)$.

10. Let $f(x) \in \mathbf{Q}[\mathbf{x}]$ be an irreducible polynomial of degree 4 and let E be the splitting field of $f(x)$. Suppose $f(x)$ has two real roots and two non-real roots ($\alpha \in \mathbf{C} \setminus \mathbf{R}$ and its complex conjugate $\bar{\alpha}$). Show that $Gal(E/\mathbf{Q})$ is not a cyclic group.