Qualifying Exam in Algebra

August,1991

Do six of the following ten problems. In this exam, **Z** denotes the integers, **Q** the rationals, and **C** the complex numbers.

1. Let G be a group, with $G \times G$ the direct product. Suppose $f : G \times G \to G$ is a homomorphism, and suppose also that there is an element n in G such that $f(n, x) = x = f(x, n)$ for all x in G. Prove that G is commutative and f is the group multiplication.

2. Prove that there are no simple groups of order p^2q, where p and q are distinct primes.

3. Let p be a prime, and let S_p be the symmetric group on $\{1, 2, ..., p\}$. Let G be a subgroup of S_p that acts transitively on $\{1, 2, ..., p\}$, and let H be a nontrivial normal subgroup of G. Prove that H also acts transitively on $\{1, 2, ..., p\}$.

4. Let $R = \mathbf{Z}[x]$ be the ring of polynomials in the indeterminate X over the integers **Z**. Determine, for each of p=2,3 and 5, whether the ideal $pR+(X^2+1)R$ is : (i) prime, (ii) maximal.

5. Let $R[X, Y]$ be the ring of polynomials in two indeterminates X, Y over an integral domain R. Prove that every unit in $R[X, Y]$ is a unit in R. Show that the result is not true when R contains nontrivial nilpotent elements.

6. Let A be a finite abelian group (written additively) with 2^7 elements. Assume that $16x = 0$ for all x in A, and A contains

8=2^3 elements x with 2x=0

32=2^5 elements x with 4x=0

64=2^6 elements x with 8x=0.

Determine the structure of the group A as a product of cyclic groups.

7. Let M be a 3×3 matrix over the rationals, and assume that $M^7 = I_3$ (the identity). Prove that $M = I_3$.

8. Let E be a finite, normal, and separable extension of a field F. If $f(X) \in F[X]$ is irreducible, prove that the irreducible factors of $f(X)$ over E all have the same degree.

9. Let E be a splitiing field of $X^{33} - 1$ over the rationals **Q**. Determine the number of subfields of E (including E and **Q**).

10. Let K be a field, and let $F = K(X, Y)$ where X and Y are algebraically independent over K. Let L be the subfield generated by K and the elements $X + X^3 + Y^{24}$ and $X + X^3 + Y^4 + Y^{24}$. Determine the degree of F over L.