Qualifying Exam in Algebra

August,1990

Do six of the following ten problems. In this exam, $\mathbf{Z}$ denotes the integers, $\mathbf{Q}$ the rationals, and $\mathbf{C}$ the complex numbers.

1. Find the order of the group $Aut(\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z})$

2. Let G be a group and let C be a subgroup of the center of G such that the group G/C is isometric to $\mathbf{Z}$. prove that C is isomorphic to $C \times \mathbf{Z}$.

3. Prove that any group of order 91 is cyclic.

4. Solve the following system for integers x, y, z:

$$x + 2y + 3z = 4, 5x + 6y + 7z = 8$$

.

5. Classify the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ over a field F up to similarity in each of the cases:

(a) char(F) $\neq 2$; (b) char(F)=2.

6. Let M be a noetherian R-module, that is, for any sequence of submodules $N_1 \subset N_2 \subset N_3 \subset ...$ there is an i so that $N_i = N_{i+1} = N_{i+2} = ...$ Let $\phi : M \to M$ be a surjective module homomorphism. Prove that ϕ is an isomorphism. *Hint.* Consider the submodules $N_k = \{x|\phi^k(x) = 0\}, k = 1, 2, 3, ...$

7. An element u of a field F is a zero of the polynomial $x^3 - 2x^2 + 3$. Find a non-zero polynomial $f(x) \in \mathbf{Z}[x]$ which vanishes at $2u - 1$.

8. A homomorphism α of the field $\mathbf{Q}(x, y)$ is given by $\alpha(x) = x + y^3, \alpha(y) = x + y + y^3$. Is α an automorphism?

9. Let $F = \mathbf{Z}/5\mathbf{Z}$, and let E be a quadratic extension of the field F. Let y be a generator of the multiplicative group of E, and let r and s be positive integers. When will $\{y^r, y^s\}$ be a basis for E (regarded as an F-vector space) over F?

10. Let $E = (\mathbf{Z}/2\mathbf{Z})(t)$ where t is transcendental over $\mathbf{Z}/2\mathbf{Z}$. Let G be the subgroup of $Aut(E)$ generated by ϕ where $\phi(t) = t + 1$, and let $F = E^G$ be the fixed field.

(a) Find $[E : F]$; (b) Find the minimal polynomial of t over F.