MATH 535 FINAL, Fall 1996

Do any five problems.

1. Let G be a group of order 153. Prove that G is abelian.

2. Let $R = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbf{F}_p\}$, with p a prime. Prove:

R is a field (with usual matrix operation) $\iff$ $p \equiv 3 \bmod 4$.

3. Let A be the finite abelian group $\mathbf{Z}_{15} \times \mathbf{Z}_{12} \times \mathbf{Z}_{10}$. How many elements are there in $2A = \{2x \mid x \in A\}$?

4. Let G be a group and A a normal abelian subgroup. Show that G/A acts on A by conjugation.

5. Let $R = \mathbf{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbf{Z}\}$, and let $I = 3R$.

(a) Prove that the ring R/I is finite.
(b) Prove or disprove: R/I is a field.

6. Prove that the matrix ring $\mathbf{M}_2(\mathbf{Z}_n)$ has only trivial (2-sided) ideals $\iff$ n is a prime.

7. Let M be a noetherian R-module, that is, for any chain of submodules $N_1 \subseteq N_2 \subseteq N_3 \subseteq ...$, there is an i such that $N_i = N_{i+1} = ...$.
Let $\phi : M \to M$ be a *surjective* module homomorphism.
Prove that ϕ is an isomorphism.

HINT: Consider the submodules $N_k = \{x \in M \mid \phi^k(x) = 0\}$.

8. Let D be a factorial domain and $I = (c) \neq 0$ a principal ideal in D. Prove: D/I contains a nilpotent element $\iff$ there exists a prime $p \in D$ with $p^2 | c$.

Note: $b \neq 0$ is nilpotent iff $b^n = 0$ for some $n \in \mathbf{N}$.