

On extensions of AF algebras

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Given a C^* -algebra A and a closed two-sided ideal J in A such that both J and A/J are AF algebras, does it follow that A is itself an AF algebra? This question was answered in the affirmative by L. Brown [1], though it was first studied in detail by G. A. Elliott [4], who reduced the problem to showing that every projection in A/J lifts to a projection in A . The proof was completed by L. Brown using the observation that the natural map $K_0(A) \rightarrow K_0(A/J)$ is surjective, which is a simple consequence of Bott periodicity. Here we present a complete proof.

Lemma 1 *Suppose J is an AF ideal in a C^* -algebra A . Then the natural map $K_0(A) \rightarrow K_0(A/J)$ is surjective.*

Proof. Since $K_1(M_n) = 0$, the continuity of K_1 implies that $K_1(B) = 0$ for any AF algebra B . Thus, $K_1(J) = 0$, and so the result follows from the six-term exact sequence. ■

Proposition 1 *Suppose J is an AF ideal in a C^* -algebra A . Then for each projection p in A/J , there is a projection P in A with $\pi(P) = p$.*

Proof. Follows from the above lemma, and a lot of work done by Brown [1] (or Effros and Elliott [3])! ■

Now, before we proceed to the main theorem, we need the following technical result [2].

Lemma 2 *Suppose J is an AF ideal in a C^* -algebra A , and let B be a finite dimensional subalgebra of A/J . Then there is a $*$ -monomorphism $\rho : B \hookrightarrow A$ such that $\pi \circ \rho = id_B$.*

Proof. Note that we can use proposition 1 to lift the minimal central projections of B to pairwise orthogonal projections in A . To do this, suppose the minimal central projections of B are p_1, p_2, \dots, p_k . First lift p_1 to P_1 , and consider $P_1^\perp J P_1^\perp$. This is an AF ideal in $P_1^\perp A P_1^\perp$, so the process can be repeated. This also reduces the problem to lifting M_n from $p_i B p_i$ to $P_i A P_i$.

Let e_{ij} be a set of matrix units for M_n . As described above, find pairwise orthogonal projections E_1, \dots, E_n such that $\pi(E_i) = e_{ii}$. For each i , let V_i be chosen so that $\pi(V_i) = e_{i1}$. Then let $Q_i = E_1 V_i^* E_i V_i E_1$. Since $\pi(Q_i) = e_{11}$, then $Q_i = E_1 - J_i$ where J_i is a positive contraction in J .

Now, since $E_1 J E_1$ is AF it has an increasing sequence of projections forming an approximate unit. Choose such a projection F with $\| (E_1 - F) J_i (E_1 - F) \| < 1/3$ for $1 \leq i \leq n$. Replacing E_1 with $E'_1 = E_1 - F$ in the definition of Q_i we get $Q'_i = E'_1 - J'_i$ with $\| J'_i \| < 1/3$.

So, the polar decomposition yields the factorization $E_i V_i E'_1 = E_{i1} Q_i'^{1/2}$ where E_{i1} is a partial isometry with initial projection E'_1 and $\pi(E_{i1}) = e_{i1}$. Also, the range projection is $E'_i = E_{i1} E_{i1}^* \leq E_i$, and $\pi(E'_i) = e_{ii}$ still.

Now, set $E_{ij} = E_{i1}E_{j1}^*$. This is a set of matrix units in A mapping onto e_{ij} in the quotient. ■

Theorem 1 *Suppose that*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{\pi} A/J \longrightarrow 0$$

is an exact sequence of C^ -algebras and that both J and A/J are AF. Then A is AF as well.*

Proof. First, note that since A/J is AF, $A/J = (\bigcup B_m)^-$ with each B_m finite dimensional. Set $A_m = \pi^{-1}(B_m)$, and so $A = \bigcup A_m^-$. Thus it suffices to show that each A_m is AF. As a result, we may assume that A/J is finite dimensional. In fact, by induction we may assume that $A/J = M_n$.

From the previous lemma, there is a $*$ -monomorphism $\rho : M_n \hookrightarrow A$ such that $\pi \circ \rho = id_{M_n}$. Let $B = \rho(M_n)$.

It will suffice to show that there is an approximate unit of projections $e_n \in J$ commuting with B . This is because if the e_n commute with B , the map $e_n J e_n \oplus B \rightarrow A$ defined by $e_n j e_n \oplus b \mapsto e_n j e_n + b(1 - e_n)$ is a $*$ -monomorphism (it is a homomorphism by commutativity, and if $e_n j e_n + b(1 - e_n) = 0$ then $b = -e_n j e_n + b e_n \in J \cap B = 0$, implying that $b = 0$ and hence $e_n j e_n = 0$ as well). The image of this map is $A_n = e_n J e_n + B$ since $e_n j e_n + b = e_n(j + e_n) + b(1 - e_n)$. Thus, for each n , A_n is an AF subalgebra of A . In fact, $A = (\bigcup A_n)^-$, since if $a \in A$, $a = j + b$ for some $j \in J$, $b \in B$ and we have $e_n j e_n \rightarrow j$. Hence A is AF.

So, we must now construct this sequential approximate unit of projections $e_n \in J$ commuting with B . Let E_{ij} be a set of matrix units for B , and P the unit of B . Now, $E_{11} J E_{11}$ is AF, so it has a sequential approximate unit of projections, say F_n . Similarly, $P^\perp J P^\perp$ does too, say F'_n . Define

$$G_n = F'_n + \sum_{i=1}^n E_{i1} F_n E_{1i}.$$

Then G_n is an approximate unit for J , and

$$\begin{aligned} E_{ij} G_n &= E_{ij} E_{j1} F_n E_{1j} \\ &= E_{i1} F_n E_{1j} \\ &= E_{i1} F_n E_{1i} E_{ij} \\ &= G_n E_{ij}. \end{aligned}$$

Hence, G_n commutes with B , and the proof is complete. ■

References

- [1] L. Brown, *Extensions of AF algebras: the projection lifting problem*, Operator Algebras and Applications **38-1** 1980 175-176.
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