Research Statement
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My research concerns existence of solutions to partial differential equations and their qualitative properties. Partial differential equations are born out of the study of natural phenomena such as wave motion, transport phenomena, and fluid flow. In the study of gas dynamics there are various models with particular properties for various flow regimes. My research focusses on both boundary value problems and initial value problems for the full, compressible Euler system. This is a system of conservation laws which models the conservation of mass, momentum and energy in gas flow. For mathematical analysis the Euler system is the most important system of conservation laws. Unlike elliptic and parabolic systems whose solutions tend to gain regularity this system is hyperbolic and solutions tend to lose regularity.

For scalar conservation laws, in one or several space dimensions, a well developed theory for existence and uniqueness of solutions exists. For systems the theory is less developed. In my research I study both 1-D and multi-D systems. For systems with more than one space dimension there is currently no general theory of existence. However, for one space dimension it has been shown that if the data are close to equilibrium, as measured in total variation, then there is a global in time unique weak solution [6].

By studying specific systems one can hope to exploit the structure of the system and gain insight for solutions with large data. The works by Nishida [10] and Nishida & Smoller [11] on the isothermal and isentropic equations, respectively, provide global existence for large bounded variation data: any bounded variation data in the case of isothermal flow, and large bounded variation data with $(\gamma-1)\times$(total variation of data) sufficiently small, in the case of isentropic flow (γ being the adiabatic gas constant). These results were extended to the full Euler system by Liu [9] and Temple [12]. More recently Temple & Young [13] have established existence for the full Euler system up to an arbitrary time for data with large total variation and sufficiently small sup-norm.

A natural question at this point for general 1-D systems is the existence of weak solutions for large data that is global in time, which is of obvious importance in real world applications. It has been demonstrated by explicit counter-examples not to hold for general systems. More precisely, Joly-Metivier-Rauch [8], Young [14] and Jenssen [7] have constructed examples (not motivated by physics) in which blow-up in sup-norm and/or total variation can occur in finite time. The hope is that the structure of the 1-D Euler system, and other physical systems, will guarantee the existence of a global in time weak solution.

1. Creating specific interaction patterns for compressible Euler system. One interaction pattern of interest is motivated by aforementioned concrete examples by Young and Jenssen for non-physical systems. If the initial data for these systems are chosen judiciously, then the solutions blow up in sup-norm and/or total variation in finite time. All known examples of this type involve wave patterns with infinitely many waves created in finite time.

In my reasearch I have shown that a similar pattern can occur for the Euler system, with carefully chosen data, but that there are only finitely many waves created in finite time [2]. Furthermore, there exists a global in time solution in this case which does not blowup in sup-norm or total variation. This indicates that the Euler system is better behaved than general systems and leads to the next problem. Namely, can the methods used be generalized to other patterns to obtain global in time solutions? I am currently working on bounds for the total variation by looking at different ways of measuring wave strengths and on finding bounds on the number of waves created up till a finite time.

2. Wave fronts. A more long term goal is to consider more general large data for the Euler system. The main mathematical issue is to produce a convergent and consistent approximation scheme which is specifically designed for the full 1-D Euler system. The front tracking algorithm is a natural candidate, but it involves certain technical issues. In particular for small data one makes use of unphysical fronts in order to limit the number of waves. For large data it seems difficult to control the strengths of fronts and therefore I would like to avoid introducing unphysical waves.

Towards this end I have been conducting an in depth analysis of all possible wave interactions involving shocks, centered rarefactions and contact discontinuities [1]. In the study I am interested in the strengths of the outgoing waves compared to that of the incoming waves. In doing this the goal is to look for appropriate ways to measure total variation and to find a priori bounds on the number of waves up to any finite time. This would be a first step in obtaining bounds on the total variation and the sup-norm in the approximation scheme. We hope to gain further insights from the calculations as well as from specific interaction patterns mentioned.

3. Multi-D shocks with symmetry. I have studied a boundary value problem for the 2-D and 3-D full compressible Euler system. I considered two concentric cylinders/spheres as the boundaries and constructed a concentric stationary shock located between the two boundaries. After constructing the stationary shock I find necessary and sufficient conditions for the existence of a stationary shock given the boundary conditions and I am able to find its location. This is part of a collaborative work with Jenssen and Williams [3, 4] in which the profile of viscous stationary solutions to the compressible Navier-Stokes equations are shown to converge to the profile of inviscid stationary shocks. The analysis applies to both the barotropic and non-barotropic flow. This is a natural extension of Gilbarg's [5] classical 1-D construction of viscous planar shock fronts and their convergence to step-shocks.

The construction of inviscid, stationary and symmetric shocks is a first step toward the more challenging problem of how dynamic shocks (with symmetry) behave in several space dimensions. A next project in this direction is to construct expanding and converging spherical shock waves and to study their interactions.

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