Math 568 Homework 8
Spring 2009
Due: Thursday, March 19

1. Marcus, Exercise 1, page 114: Show that $E(Q|\mathfrak{p})$ is a normal subgroup of $D(Q|\mathfrak{p})$ directly from the definition of these groups.

2. Marcus, Exercise 5, (a)–(c), page 115: Let L/K be an extension of number fields such that L is Galois over K with Galois group G. Let $\mathfrak{p}$ be a prime of K. By "intermediate field" we will mean "intermediate field different from K and L".

   (a) Prove that if $\mathfrak{p}$ is inert in L then G is cyclic.

   (b) Suppose $\mathfrak{p}$ is totally ramified in every intermediate field, but not totally ramified in L. Prove that no intermediate fields can exist, hence G is cyclic of prime order. (*Hint*: inertia field.)

   (c) Suppose every intermediate field contains a unique prime lying over $\mathfrak{p}$, but L does not. Prove the same as in (b). (*Hint:* decomposition field.)

3. Marcus, Exercise 5, (d)–(e), page 115: Let L/K be an extension of number fields such that L is Galois over K with Galois group G. Let $\mathfrak{p}$ be a prime of K. By "intermediate field" we will mean "intermediate field different from K and L".

   (a) Suppose $\mathfrak{p}$ is unramified in every intermediate field, but ramified in L. Prove that G has a unique smallest nontrivial subgroup H, and that H is normal in G; use this to show that G has prime power order, H has prime order, and H is contained in the center of G.

   (b) Suppose $\mathfrak{p}$ splits completely in every intermediate field, but not in L. Prove the same as in (a). Find an example of this over $\mathbb{Q}$.

4. Marcus, Exercise 6, (a)–(b), page 116: Let m, n be distinct square-free integers $\neq 1$. The biquadratic field $K := \mathbb{Q}(\sqrt{m}, \sqrt{n})$ is a Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus there are three quadratic subfields of K (what is the third?). Let p be a prime in $\mathbb{Z}$.

   (a) Suppose p is ramified in each of the quadratic subfields. What happens in K? Find an example.

   (b) Suppose p splits completely in each of the quadratic subfields. What happens in K? Find an example.