Math 568 Homework 6
Spring 2009
Due: Thursday, February 26

1. Marcus, page 84, Exercise 13.

2. Let d be a squarefree integer and p a prime number not dividing $2d$. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$. Show that $(p) = p\mathcal{O}$ is a prime ideal of $\mathcal{O}$ if and only if the congruence $x^2 \equiv d \pmod{p}$ has no solution in rational integers.

3. Show that $\mathbb{Z}[\sqrt{-5}]$ has exactly two ideal classes, the class of principal ideals and the class containing $(2, 1+\sqrt{-5})$.

4. Let $\mathfrak{m}$ be a nonzero ideal of the Dedekind domain R. Show that in every ideal class of R, there exists an ideal which is prime to $\mathfrak{m}$.

5. Let L/K be an extension of number fields with corresponding rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$. Suppose that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$. Let $f(x)$ be the minimal polynomial of α over K. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_K$. Let $g_1(x), \ldots, g_r(x)$ be monic polynomials in $\mathcal{O}_K[x]$ that are distinct and irreducible modulo $\mathfrak{p}$, and such that $f(x) \equiv \prod g_i(x)^{e_i}$ modulo $\mathfrak{p}$. Show that
$$\mathfrak{p}\mathcal{O}_L = \prod (\mathfrak{p}, g_i(\alpha))^{e_i}$$
is the factorization of $\mathfrak{p}\mathcal{O}_L$ into a product of powers of distinct prime ideals. Show that the residue field degree f_i is equal to the degree of g_i.