Math 568 Homework 5
Spring 2009
Due: Thursday, February 19

1. Show directly that $R = \mathbb{Z}[\sqrt{-7}]$ does not have unique factorization of ideals by finding an ideal that cannot be factored uniquely into products of prime ideals.

2. Let A be an integral domain with only finitely many prime ideals. Show that A is a Dedekind domain if and only if it is a principal ideal domain.

3. Let K be any field and let L be a finite separable extension of the field $K(x)$ of rational functions over K. Prove that the integral closure of $K[x]$ in L is a Dedekind domain. (Hint: imitate the proof that the ring of integers $\mathcal{O}_F$ inside a number field F is a Dedekind domain.)

4. Prove that every ideal of a Dedekind domain R can be generated by two elements.

5. Let k be a field, and let A be the subring $k[X^2, X^3]$ of $k[X]$.

   (a) Show that $k[X]$ is a finitely generated $k[X^2]$-module. Use this to show that A is a Noetherian ring.

   (b) Show that every nonzero prime ideal of A is maximal, but that A is not a Dedekind domain.

   Hence A satisfies conditions (1) and (2) for a Dedekind domain, but not (3).