Math 568 Homework 4
Spring 2009
Due: Thursday, February 12

1. Show that the ring of integers in $\mathbb{Q}[\cos(2\pi/m)]$ is $\mathbb{Z}[2\cos(2\pi/m)]$.

2. (a) Let K be an algebraic number field. Show that the sign of the discriminant $\operatorname{disc}(K)$ is $(-1)^s$, where $2s$ is the number of homomorphisms $K \hookrightarrow C$ whose image is not contained in $\mathbb{R}$.

   (b) Show that $X^3 - 3X + 1$ is an irreducible polynomial in $\mathbb{Q}[X]$ with three real roots. Let α be one of them and let $K = \mathbb{Q}[\alpha]$. Compute $\operatorname{disc}(\mathbb{Z}[\alpha])$ and deduce that

   $$\mathcal{O}_K \supset \mathbb{Z}[\alpha] \supset 3^m \mathcal{O}_K$$

   for some m.

3. Let K be an algebraic number field. Prove that $\operatorname{disc}(K) \equiv 0$ or $1 \pmod 4$.

4. Let $f(x) = x^3 + ax + b$ with $a, b \in \mathbb{Z}$, and assume f is irreducible over $\mathbb{Q}$. Let α be a root of f.

   (a) Show that $f'(\alpha) = -(2a\alpha + 3b)/\alpha$.

   (b) Show that $2a\alpha + 3b$ is a root of $\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b$.
   Use this to find $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b)$.

   (c) Show that $\operatorname{disc}(\alpha) = -(4a^3 + 27b^2)$.

5. Marcus, exercise 34, page 47.