# Math 568 Homework 3
# Spring 2009
# Due: Thursday, February 5

1. Let $K$ be a number field. Show that for any $\alpha \in K$ there exists $m \in \mathbb{Z}, m \neq 0$, such that $m\alpha$ is an algebraic integer.

2. Show that $\left\{1, \sqrt[3]{2}, \sqrt[3]{2}^2\right\}$ is an integral basis of $\mathbb{Q}(\sqrt[3]{2})$.

3. (a) Let $\alpha$ be a root of $X^3 - X - 1$. Show that $\{1, \alpha, \alpha^2\}$ is an integral basis for $\mathbb{Q}[\alpha]$ (and $\mathbb{Z}[\alpha]$ is the ring of integers in $\mathbb{Q}[\alpha]$).

   (b) Consider the field $\mathbb{Q}[\alpha]$ where $\alpha$ is a root of $f(X) = X^5 - X - 1$. Show that the ring of integers in $\mathbb{Q}[\alpha]$ is $\mathbb{Z}[\alpha]$.

4. Show that a root of a monic polynomial with algebraic integer coefficients is an algebraic integer.

5. For $n > 1$ an integer, prove that $2\cos(2\pi/n)$ and $2\sin(2\pi/n)$ are algebraic integers. Show that for $n \geq 5$, $\cos(2\pi/n)$ and $\sin(2\pi/n)$ are *not* algebraic integers. (Hint: recall that $e^{i\theta} = \cos\theta + i\sin\theta$.)

6. Show that the ring of integers $\mathcal{O}_L$ of a number field $L$ is the largest subring of $L$ that is finitely generated as a $\mathbb{Z}$-module.