Math 568 Homework 10
Spring 2009
Due: Thursday, April 2

1. (5 points) Marcus, Exercise 15, page 120: Let F be any field. The set V of all functions from F to F is a vector space over F with the obvious pointwise operations. Prove that distinct automorphisms $\sigma_1, \ldots, \sigma_n$ of F are always linearly independent over F. (See also the hint for this exercise in Marcus.)

2. (10 points) Marcus, Exercise 16, page 120: Let K be a number field. Suppose $p \in \mathbb{Z}$ is a prime such that $d = \mathrm{disc}(K)$ is exactly divisible by p^m, m odd ($p^m \mid d$, $p^{m+1} \nmid d$). Prove that p is ramified in K by considering the fact that the normal closure of K over $\mathbb{Q}$ contains $\mathbb{Q}[\sqrt{d}]$. Show that $\mathrm{disc}(K)$ can be replaced by $\mathrm{disc}(\alpha_1, \ldots, \alpha_n)$ for any $\alpha_1, \ldots, \alpha_n \in \mathcal{O}_K$ such that $\alpha_1, \ldots, \alpha_n$ is a basis for K over $\mathbb{Q}$.

3. (5 points) Marcus, Exercise 23, page 123: Let V_1 be the ramification group defined in Problem 3 on Homework 9, and use the same notation as in this problem. Show that V_1 is the Sylow p-subgroup of E, where p is the prime of $\mathbb{Z}$ lying under Q. Conclude that V_1 is nontrivial iff $e(Q|\mathfrak{p})$ is divisible by p. (You may use the results from earlier exercises in Marcus that prove properties of these ramification groups.)

4. (5 points) Marcus, Exercise 31a-b, page 125: Let K be an abelian extension of $\mathbb{Q}$ with $[K : \mathbb{Q}] = p^m$. Suppose that q is a prime $\neq p$ which is ramified in K. Fix a prime Q of K lying over q, and let $e := e(Q|q)$.

   (a) Prove that $V_1(Q|q) = \{1\}$.

   (b) Prove that $e(Q|q)$ divides $q - 1$. (See Marcus, exercise 26, page 124. You may use the results from this exercise.)