

**Math 536 Homework 9**  
**Spring 2008**  
**Due: Friday, March 28**

1. Let  $R$  be a commutative ring with identity  $1 \neq 0$ . Show that  $R^m \cong R^n$  implies  $m = n$ . (I.e., for commutative rings  $R$  with identity, any finitely generated free module  $R$ -module has a well defined rank.)
2. Show that for a noncommutative ring  $R$ , we can have  $R^m \cong R^n$  for distinct integers  $m, n$ . (Hint: Let  $k$  be a field and let  $V$  be the polynomial ring in one variable over  $k$ . Then  $V$  is an infinite-dimensional vector space over  $k$ . Let  $R$  be the endomorphism ring of  $V$ ,  $R = \text{End}_k(V)$ , and show that  $R$  has the desired property.)
3. Suppose that  $R$  is a commutative ring with identity such that every submodule of every free  $R$ -module is free. Show that  $R$  is a PID.
4. Let  $R := \mathbb{Z}[X]$ . Give an example of a finitely generated  $R$ -module that does not decompose into a finite direct sum of cyclic  $R$ -modules.
5. Let  $R$  be a domain containing a field  $k$  as a subring. Suppose that  $R$  is a finite dimensional vector space over  $k$  under the ring multiplication. Show that  $R$  is a field.