Math 536 Homework 6
Spring 2008
Due: Friday, February 29

In all of the following problems R will denote a commutative ring with identity.

1. Let S be a multiplicative subset of R not containing 0. Let $\mathfrak{p}$ be a maximal element in the set of ideals of R whose intersection with S is empty. Show that $\mathfrak{p}$ is prime.

2. Assume that R is a domain, and let $\mathfrak{p}$ be a prime ideal of R. Let $S := R - \mathfrak{p}$. Show that S is a multiplicative subset of R. Let $R_{\mathfrak{p}} := S^{-1}R$. Show that the ring $R_{\mathfrak{p}}$ has a unique maximal ideal, consisting of all elements a/s with $a \in \mathfrak{p}$ and $s \notin \mathfrak{p}$.

3. Show that for any domain R, the prime ideals of $S^{-1}R$ are in bijection with the prime ideals of R which do not intersect S. (In fact, this is true for any ring R.)

4. A ring R is called a *local ring* if it is commutative and has a unique maximal ideal.

   Let $f : A \to A'$ be a surjective homomorphism of rings (with identity), and assume that A is local, $A' \neq 0$. Show that A' is local.

5. Let D be an integer ≥ 1 and let R be the set of all elements $a + b\sqrt{-D}$ with $a, b \in \mathbb{Z}$.

   (a) Show that R is a ring.

   (b) Let $N : R \to \mathbb{Z}$ be the norm map, i.e. the map given by

   $$N(a + b\sqrt{-D}) = (a + b\sqrt{-D})(a - b\sqrt{-D}).$$

   Show that for $u, v \in R$ we have $N(uv) = N(u)N(v)$.

   (c) Show that $u \in R$ is a unit if and only if $N(u) = \pm 1$.

   (d) Show that if $D \geq 2$, then the only units in R are ± 1.

   (e) Show that $3, 2 + \sqrt{-5}, 2 - \sqrt{-5}$ are irreducible elements in $\mathbb{Z}[\sqrt{-5}]$.

   (f) Use the above elements to prove that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

   (g) Are the elements $3, 2 + \sqrt{-5}, 2 - \sqrt{-5}$ prime in the ring R?