Math 536 Homework 12
Spring 2008
Due: Monday, April 28

1. (25 points) What is the Galois group of the splitting field of each of the following polynomials?

   (a) $X^3 - X - 1$ over $\mathbb{Q}$.

   (b) $X^3 - 10$ over $\mathbb{Q}$.

   (c) $X^3 - 10$ over $\mathbb{Q}(\sqrt{2})$.

   (d) $X^3 - 10$ over $\mathbb{Q}(\sqrt{-3})$.

   (e) $X^3 - X - 1$ over $\mathbb{Q}(\sqrt{-23})$.

2. (10 points) Compute the Galois group of a splitting field for $X^5 - 2$ over $\mathbb{Q}$.

3. (20 points) Let p be an odd prime, and let ζ be a primitive p-th root of unity in $\mathbb{C}$ (for example, take $\zeta = e^{2\pi i/p}$). Let $E = \mathbb{Q}[\zeta]$, and let $G =$Gal$(E/\mathbb{Q})$. Show that $G = (\mathbb{Z}/p\mathbb{Z})^*$. Let H be the subgroup of index 2 in G. Put $\alpha := \sum_{i \in H} \zeta^i$ and $\beta := \sum_{i \in G-H} \zeta^i$. Show:

   (a) α and β are fixed by H;

   (b) if $\sigma \in G - H$, then $\sigma\alpha = \beta, \sigma\beta = \alpha$.

   Use (a) and (b) to show that α and β are roots of the polynomial $X^2 + X + \alpha\beta \in \mathbb{Q}[X]$. Compute $\alpha\beta$ and show that the fixed field of H is $\mathbb{Q}[\sqrt{p}]$ when $p \equiv 1 \mod 4$ and $\mathbb{Q}[\sqrt{-p}]$ when $p \equiv 3 \mod 4$.

4. (20 points) Let $M = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$ and $E = M[\sqrt{(\sqrt{2}+2)(\sqrt{3}+3)}]$ (subfields of $\mathbb{R}$).

   (a) Show that M is Galois over $\mathbb{Q}$ with Galois group the Klein 4-group $C_2 \times C_2$.

   (b) Show that E is Galois over $\mathbb{Q}$ with Galois group the quaternion group.