Math 497A Homework 7
Fall 2008
Due: Friday, October 24

(1) For a prime p let C_p be the elliptic curve with equation $C_p : y^2 = x^3 + px$. Show that the rank of $C_p(\mathbb{Q})$ is 0,1, or 2.

(2) (a) Let p be a prime which is congruent to 2 modulo 3, and let $c \in \mathbb{F}_p^*$. Prove that the curve

$$E : y^2 = x^3 + c$$

satisfies $\#E(\mathbb{F}_p) = p + 1$.

(b) Let c be a non-zero integer which is sixth power free. (I.e., there are no primes p with $p^6 \mid c$.) Let E be the elliptic curve

$$E : y^2 = x^3 + c.$$

Show that $E_{\text{tor}}(\mathbb{Q})$ is isomorphic to

(i) $\mathbb{Z}/6\mathbb{Z}$ if c=1;

(ii) $\mathbb{Z}/3\mathbb{Z}$ if $c \neq 1$ is a square, or if $c = -432$;

(iii) $\mathbb{Z}/2\mathbb{Z}$ if $c \neq 1$ is a cube;

(iv) $\{\mathcal{O}\}$ otherwise.

(3) Let E be the elliptic curve given by

$$E : y^2 = x^3 - 4x^2 + 16.$$

(a) Let $M_p = \#E(\mathbb{F}_p)$ be the number of points on E over the field $\mathbb{F}_p$. (This includes $\mathcal{O}$.) Calculate M_p for all primes $3 \leq p \leq 13$. (If you use Maple to do this, you should calculate M_p for more values. This will help with the later parts.)

(b) Let $F(q)$ be the formal power series given by the infinite product

$$F(q) = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 = q - 2q^2 - q^3 + 2q^4 + \dots.$$

Let N_n be the coefficient of q^n in $F(q)$,

$$F(q) = \sum_{n=1}^{\infty} N_n q^n.$$

Calculate N_n for $n \leq 13$. (Again, if you use Maple, compute N_n for more values.)

(c) For each prime p as in parts (a) and (b), compute $M_p + N_p$. Formulate a conjecture as to what this value should be in general.

(d) (**) Prove that your conjecture is correct.

(4) Compute the rank of the elliptic curve $E : y^2 = x^3 - 82x$.