

# Some Fine Combinatorics

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Dedicated to George Andrews on the occasion of his 70th birthday

## Abstract

In 1988, N. J. Fine published a monograph entitled *Basic Hypergeometric Series and Applications* in which he proved a number of results concerning the series  $F(a, b; t : q)$ . In this paper, we present a new combinatorial interpretation for the series  $F(a, b; t : q)$  and use Fine's work as a guide for proving the Rogers-Fine identity and many of its properties in this setting.

*Keywords:* Rogers-Fine identity, basic hypergeometric series,  $q$ -difference equations

*Mathematics Subject Classification:* 05A15

## 1 Introduction

N. J. Fine [2] presented an introduction to the theory of basic hypergeometric series by giving an extensive study of the series

$$F(a, b; t : q) = \sum_{n=0}^{\infty} \frac{(aq)_n t^n}{(bq)_n}$$

where  $(z)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$ . By deriving numerous functional equations satisfied by  $F(a, b; t : q)$ , one of the many results Fine proved is the so-called Rogers-Fine identity [3]

$$F(a, b; t : q) = \sum_{n=0}^{\infty} \frac{(aq)_n (atq/b)_n (1-atq^{2n+1}) b^n t^n q^{n^2}}{(bq)_n (t)_{n+1}} \quad (1)$$

along with numerous well-known identities that appear as special cases of (1). It is our goal to provide a new combinatorial context for  $F(a, b; t : q)$  that easily explains the Rogers-Fine identity, including many of its properties and applications.

To this end, we consider weighted permutations of a multiset that, for illustrative purposes, we present as tilings of a single row of finite length using black, gray, and white squares. Given a tile,  $t$ , we define its weight,  $w(t)$ , as

$$w(t) = \begin{cases} aq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \text{ } \square \text{ or } \square \text{ to its left} \\ bq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \text{ } \square \text{ or } \square \text{ to its left} \\ c & \text{if } t \text{ is a } \square. \end{cases}$$

The weight of a tiling  $T$  is given by

$$w_T(a, b, c; q) = \prod_{t \in T} w(t).$$

For example, if the tiling  $T$  is given by



then the weight of  $T$  is

$$\begin{aligned} w_T(a, b, c; q) &= c \cdot aq \cdot c \cdot bq^2 \cdot c \cdot c \cdot c \cdot bq^6 \cdot aq^7 \cdot c \cdot c \cdot bq^9 \cdot c \cdot aq^{11} \\ &= a^3 b^3 c^8 q^{36}. \end{aligned}$$

Clearly the power of  $a$  in  $w_T(a, b, c; q)$  keeps track of the number of black squares in  $T$ , the power of  $b$  keeps track of the number of gray squares, and the power of  $c$  keeps track of the number of white squares. The power of  $q$  is very similar to the usual inversion statistic of a permutation, except that it also includes the number of pairs of gray squares.

For the purposes of this paper, we will restrict ourselves to considering tilings that do not end with a white square. In particular, define

$$G(a, b, c; q) = \sum_T w_T(a, b, c; q)$$

where the sum is over all tilings of finite length that end with a black or gray square, including the empty tiling, which has weight 1. Note that we are considering all tilings that do not end with a white square and not restricting ourselves to tilings of a certain length.

It should be clear that  $aG(a, b, c; q)$  is the generating function for tilings that start with a black square and do not end with a white square and that  $c^j G(a, b, c; q)$  is the generating function for tilings that end with exactly  $j$  consecutive white squares. Consequently,

$$(1 - a)G(a, b, c; q)$$

can be interpreted as the generating function for tilings that do not start with a black square and do not end with a white square and

$$\frac{G(a, b, c; q)}{1 - c}$$

is the generating function for all tilings without restriction. While we could have chosen either of these types of tilings as our focal point, as it turns out, tilings that do not end with a white square will be in keeping with the identities as presented in [2].

## 2 Proof of the Rogers-Fine Identity

We begin by making the substitutions  $a \rightarrow -\frac{b}{aq}$ ,  $b \rightarrow c$  and  $t \rightarrow a$  in (1), which results in

$$\sum_{n=0}^{\infty} \frac{(-b/a)_n a^n}{(cq)_n} = \sum_{n=0}^{\infty} \frac{(-b/a)_n (-b/c)_n a^n c^n q^{n^2} (1 + bq^{2n})}{(a)_{n+1} (cq)_n}.$$

We will show that both sides of this transformed identity are equal to  $G(a, b, c; q)$ .

Consider all tilings that do not end with a white square and have exactly  $n$  black or gray squares. Each such tiling can be broken up into  $n$  segments consisting of consecutive white squares followed by a single black or gray square. In other words, each black or gray square marks the end of a segment. The example from the previous section would be broken up in the following manner.



Suppose that the  $i$ th segment from left to right consists of exactly  $j \geq 0$  white squares followed by a single tile  $t$ , which is a black or gray square. Note that each white square in this segment has exactly  $i - 1$  black or gray squares to its left and  $n + 1 - i$  black or gray squares to its right. Therefore, each white square contributes a factor of  $cq^{n+1-i}$  to the weight of the tiling since each of the last  $n + 1 - i$  black or gray squares will count the white square as being to its left. Tile  $t$  contributes a factor of  $a$  if it is a black square or a factor of  $bq^{n-i}$  if it is a gray square since each of the last  $n - i$  black or gray squares will count tile  $t$  as being to its left. Allowing for all possible values of  $j$  and  $t$ , the  $i$ th segment of tiles accounts for a weight of

$$(a + bq^{n-i}) \sum_{j=0}^{\infty} c^j q^{j(n+1-i)} = \frac{a + bq^{n-i}}{1 - cq^{n+1-i}}.$$

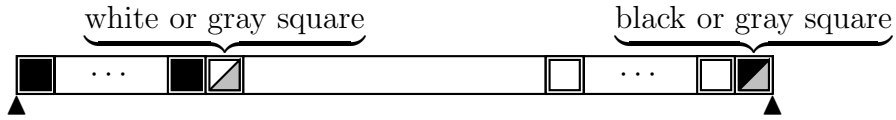
Therefore, the generating function for all tilings that have exactly  $n$  black or gray squares is given by

$$\prod_{i=1}^n \frac{a + bq^{n-i}}{1 - cq^{n+1-i}} = \frac{(-b/a)_n a^n}{(cq)_n}.$$

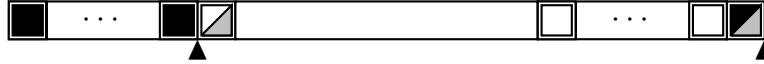
Summing over all  $n \geq 0$  accounts for all tilings that do not end with a white square and yields

$$G(a, b, c; q) = \sum_{n=0}^{\infty} \frac{(-b/a)_n a^n}{(cq)_n}.$$

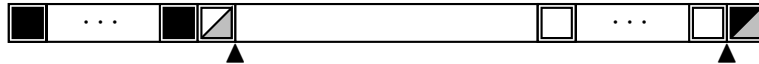
To complete the proof of the Rogers-Fine identity, we need to construct the same set of tilings in a different manner. To this end, we define the *weighted center* of a tiling to be the place on the board where the number of gray or white squares to its left is the same as the number of black or gray squares to its right. To justify the existence of such a position, the weighted center of a tiling can be found in the following manner. Initially place one marker ( $\blacktriangle$ ) to the left of the first tile and another marker to the right of the last tile. Then alternate applying the following two steps, until a weighted center is found.



**Step 1:** If the left marker has a black square immediately to its right, then repeatedly move the left marker one tile to its right until this is no longer the case. At the same time, if the right marker has a white square immediately to its left, then repeatedly move the right marker one tile to its left until this is no longer the case.



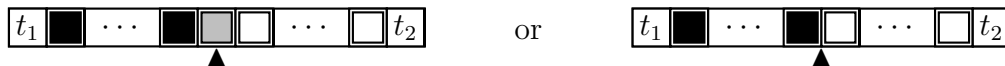
**Step 2:** If the two markers are now in the same location, then that location is the weighted center. If there is exactly one tile separating the two markers, then that tile must be a gray square, and the middle of that gray square is the weighted center. Otherwise, move the left marker one tile to its right, move the right marker one tile to its left, and repeat Step 1.



Note that after either step, the number of white or gray squares to the left of the left marker is always equal to the number of black or gray squares to the right of the right marker. Thus when the process terminates, we have located a position on the board that satisfies the definition of a weighted center. And each time the process repeats, the markers are closer together than when they started and therefore the process must terminate.

Now suppose that there are two places on the board that satisfy the definition of a weighted center. Let the two positions be indicated by  $c_1$  and  $c_2$  with  $c_1$  being to the left of  $c_2$ . If there are  $x_i$  white or gray squares strictly to the left of  $c_i$  for  $i = 1, 2$ , then  $x_1 \leq x_2$  since  $c_2$  is to the right of  $c_1$ . However, we must also have  $x_1 \geq x_2$ , since  $x_i$  is also the number of black or gray squares to the right of  $c_i$  and  $c_1$  is to the left of  $c_2$ . Thus  $x_1 = x_2$ .

Next, let tile  $t_1$  be the  $x_1$ th white or gray square from the left and let tile  $t_2$  be the  $x_1$ th black or gray square from the right. Since the weighted center must be between these two tiles, the only tiles separating  $t_1$  and  $t_2$  could be a sequence of black squares, followed by at most one gray square, followed by a sequence of white squares.



However, the weighted center for this portion of the tiling is unique (indicated by a single marker in each of the above diagrams) and thus,  $c_1$  and  $c_2$  must mark the same location on the board. In other words, every tiling must have a *unique* weighted center.

We say that a tiling is of *degree*  $n$  if there are exactly  $n$  gray or white squares strictly to the left (or equivalently,  $n$  black or gray squares strictly to the right) of its weighted center. If the

weighted center falls on a gray square, that gray square is not counted as part of the degree of the tiling. For example, the following tiling is of degree 4 and the weighted center is indicated by a single marker.



Notice that inserting any number of black squares before the weighted center and/or white squares after the weighted center does not change the degree of the tiling.



Furthermore, if the weighted center is between two tiles, then inserting a gray square at the weighted center does not change the degree of the tiling, however, the weighted center of the resulting tiling would coincide with the inserted tile.



Now back to our proof. Consider all tilings of degree  $n$ . To construct such a tiling, first place  $n$  gray and/or white squares in positions 1 through  $n$ . A white square placed in position  $j$  contributes a factor of  $c$  to the weight of the tiling. A gray square in position  $j$  contributes a factor of  $bq^{j-1}$  since there are exactly  $j - 1$  gray or white squares to its left. In other words, the factor of  $c + bq^{j-1}$  represents the choice of initially placing a white or gray square, respectively, in position  $j$  for  $1 \leq j \leq n$ . Therefore, allowing for all possible arrangements of gray and white squares in the first  $n$  positions contributes

$$\prod_{j=1}^n (c + bq^{j-1}) = (-b/c)_n c^n$$

to the weight of all tilings of degree  $n$ .

Next, place  $n$  black and/or gray squares in positions  $n + 1$  through  $2n$ . A black square placed in position  $n + j$  contributes a factor of  $aq^n$  to the weight of the tiling. A gray square placed in position  $n + j$  contributes a factor of  $bq^{2n-j}$  since the  $n$  squares in positions 1 through  $n$  are to its left and each of the last  $n - j$  black or gray squares are to its right. In other words, the factor of  $aq^n + bq^{2n-j}$  represents the choice of initially placing a black or gray square, respectively, in position  $n + j$  for  $1 \leq j \leq n$ . Therefore, allowing for all possible arrangements of black and gray squares contributes

$$\prod_{j=1}^n (aq^n + bq^{2n-j}) = (-b/a)_n a^n q^{n^2}$$

to the weight of all tilings of degree  $n$ .

Now decide whether or not the weighted center coincides with the position of a gray square. Or equivalently, whether or not to insert a gray square immediately after the  $n$ th square. If you do not insert a gray square, then the weight of the tiling is unchanged. However, if a gray square is inserted, the weight of the tiling changes but the degree does not. In particular, the weight of the gray square is  $bq^n$  and its presence increases the weight of each of the  $n$  black or gray squares that appear to its right by a factor of  $q$ . Therefore, if there is a gray square at the weighted center, then it increases the weight of the tiling by a factor of  $bq^{2n}$ . Thus the factor

$$(1 + bq^{2n})$$

represents the choice of having a gray square at the weighted center or not.

At this point, we have guaranteed that the tiling is of degree  $n$ , but we have not yet constructed all such tilings. Recall that inserting any number of black squares before the weighted center and/or any number of white squares after the weighted center does not change the degree of the tiling.

Suppose that exactly  $j \geq 0$  consecutive black squares are inserted immediately *after* the  $i$ th gray or white square for  $i = 0, 1, 2, \dots, n$ , where the  $i = 0$  case means that the black squares are placed *before* the first gray or white square. Each of the  $j$  black squares contributes a factor of  $aq^i$  to the weight of the tiling since there are exactly  $i$  gray or white squares to its left. Allowing for all possible values of  $j$ , the consecutive black squares immediately after the  $i$ th gray or white square contributes

$$\sum_{j=0}^n a^j q^{ji} = \frac{1}{1 - aq^i}$$

to the weight of all tilings of degree  $n$ .

Now suppose exactly  $j \geq 0$  consecutive white squares are inserted immediately before the  $i$ th black or gray square to the right of the weighted center for  $i = 1, 2, \dots, n$ . Each of the  $j$  white squares contributes a factor of  $cq^{n+1-i}$  to the weight of the tiling since there are exactly  $n + 1 - i$  black or gray squares to its right. Allowing for all possible values of  $j$ , the consecutive white squares immediately before the  $i$ th black or gray square after the weighted center contributes

$$\sum_{j=0}^n c^j q^{j(n+1-i)} = \frac{1}{1 - cq^{n+1-i}}$$

to the weight of all tilings of degree  $n$ . Therefore, the generating function for all tilings of degree  $n$  is given by

$$\frac{(-b/a)_n (-b/c)_n a^n c^n q^{n^2} (1 + bq^{2n})}{(1 - a) \cdots (1 - aq^n) (1 - cq) \cdots (1 - cq^n)} = \frac{(-b/a)_n (-b/c)_n a^n c^n q^{n^2} (1 + bq^{2n})}{(a)_{n+1} (cq)_n}.$$

Summing over all  $n \geq 0$  again accounts for all tilings that do not end with a white square and yields

$$G(a, b, c; q) = \sum_{n=0}^{\infty} \frac{(-b/a)_n (-b/c)_n a^n c^n q^{n^2} (1 + bq^{2n})}{(a)_{n+1} (cq)_n}$$

which completes the proof of the Rogers-Fine identity.

### 3 Functional Equations

In sections 2 and 4 of [2], Fine presented a number of functional equations satisfied by  $F(a, b; t : q)$ , one of which was used to prove the Rogers-Fine identity. For the sake of completeness, we present here many of these functional equations in the context of the series  $G(a, b, c; q)$ .

For each of the functional equations presented in this section, we will make use of the following trivial observation regarding the substitution  $a \rightarrow aq$ . If tiling  $T$  contains exactly  $n$  black squares, then

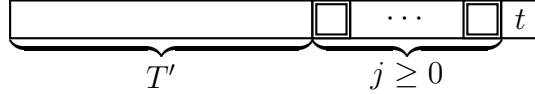
$$w_T(aq, b, c; q) = q^n w_T(a, b, c; q).$$

In other words, to increase the weight of the tiling by a factor of  $q$  for each black square, simply replace  $a$  with  $aq$ . In a similar manner, the substitutions  $b \rightarrow bq$  and  $c \rightarrow cq$  can be used to increase the weight of a tiling by the number of gray and white squares, respectively. With that in mind, we are now ready to prove the following identities.

#### Theorem 1

$$G(a, b, c; q) = 1 + \frac{a + b}{1 - cq} G(a, bq, cq; q)$$

**Proof.** Since the weight of the empty tiling is 1,  $G(a, b, c; q) - 1$  is the generating function for all weighted tilings that have at least one black or gray square. Any such tiling,  $T$ , can be uniquely decomposed into a (possibly empty) tiling,  $T'$ , followed by a sequence of  $j \geq 0$  consecutive white squares, followed by a single black or gray square,  $t$ .



The power of  $q$  in  $w(t)$  is  $j$  plus the number of gray or white squares in  $T'$ , which is determined by replacing  $b$  with  $bq$  and  $c$  with  $cq$  in  $w_{T'}(a, b, c; q)$ . In other words, the weight of  $T$  is given by

$$w_T(a, b, c; q) = \begin{cases} ac^j q^j w_{T'}(a, bq, cq; q) & \text{if } t \text{ is a } \blacksquare \\ bc^j q^j w_{T'}(a, bq, cq; q) & \text{if } t \text{ is a } \blacksquare \end{cases}$$

Summing over all possible tilings  $T'$  and nonnegative integers  $j$  yields

$$\begin{aligned} G(a, b, c; q) - 1 &= \sum_{j=0}^{\infty} \sum_{T'} ac^j q^j w_{T'}(a, bq, cq; q) + bc^j q^j w_{T'}(a, bq, cq; q) \\ &= (a + b) \sum_{j \geq 0} c^j q^j \sum_{T'} w_{T'}(a, bq, cq; q) \\ &= \frac{a + b}{1 - cq} G(a, bq, cq; q) \end{aligned}$$

as desired. □

#### Theorem 2

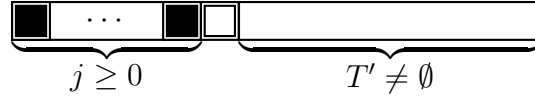
$$G(a, b, c; q) = \frac{1 - c}{1 - a} + \frac{b + c}{1 - a} G(aq, bq, c; q)$$

**Proof.** We begin our proof by pointing out that any tiling  $T$  falls into exactly one of the following three categories based on the first non-black tile, if any, to appear in  $T$ .

If all of the tiles in  $T$  are black squares, then suppose that  $T$  consists of exactly  $j \geq 0$  consecutive black squares and no other tiles. Since each square contributes a factor of  $a$  to the weight of  $T$ , the generating function for all such tilings, which includes the empty tiling, is given by

$$\sum_{j=0}^{\infty} a^j = \frac{1}{1-a}.$$

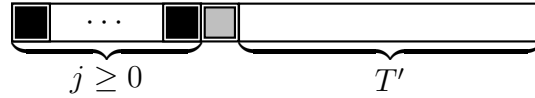
If the first non-black tile is a white square, then  $T$  can be decomposed into  $j \geq 0$  consecutive black squares followed by a single white square followed by a nonempty tiling,  $T'$ , since  $T$  cannot end with a white square.



The first white square contributes a factor of  $c$  as well as one factor of  $q$  for each black or gray square in  $T'$ . These additional factors of  $q$  can be obtained by replacing  $a$  with  $aq$  and  $b$  with  $bq$  in  $w_{T'}(a, b, c; q)$ . Therefore, the generating function for all such  $T$  is given by

$$\sum_{j=0}^{\infty} \sum_{T' \neq \emptyset} ca^j w_{T'}(aq, bq, c; q) = \frac{c}{1-a} [G(aq, bq, c; q) - 1].$$

Otherwise, the first non-black tile is a gray square, in which case  $T$  can be decomposed into  $j \geq 0$  consecutive black squares followed by a single gray square followed by any tiling,  $T'$ .



$$\sum_{j=0}^{\infty} \sum_{T'} ba^j w_{T'}(aq, bq, c; q) = \frac{b}{1-a} G(aq, bq, c; q)$$

and

$$\begin{aligned} G(a, b, c; q) &= \frac{1}{1-a} + \frac{c}{1-a} [G(aq, bq, c; q) - 1] + \frac{b}{1-a} G(aq, bq, c; q) \\ &= \frac{1-c}{1-a} + \frac{b+c}{1-a} G(aq, bq, c; q) \end{aligned}$$

as claimed. □

### Theorem 3

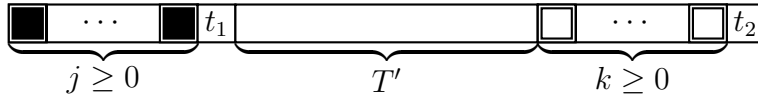
$$G(a, b, c; q) = \frac{1+b}{1-a} + \frac{(a+b)(b+c)q}{(1-a)(1-cq)} G(aq, bq^2, cq; q)$$

**Proof.** For this proof, we will break up tilings into two categories, based on whether or not the tiling is of degree 0.

Consider all tilings of degree 0. Each such tiling consists of exactly  $j \geq 0$  consecutive black squares and no other tiles or a sequence of exactly  $j \geq 0$  consecutive black squares followed by a single gray square. The generating function for all such tilings is given by

$$\sum_{j=0}^{\infty} a^j + a^j b = \frac{1+b}{1-a}.$$

Now consider all tilings of degree at least 1. Any such tiling consists of  $j \geq 0$  consecutive black squares, followed by a single gray or white square,  $t_1$ , followed by a tiling  $T'$  of degree at least 0 (i.e., any tiling), followed by  $k \geq 0$  consecutive white squares, followed by a single black or gray square,  $t_2$ .



The tile  $t_1$  contributes a factor of  $q$  to the weight of  $T$  for each black or gray square in  $T'$ . The tile  $t_2$  contributes a factor of  $q$  for each gray or white square in  $T'$ . In other words, these additional factors of  $q$  can be obtained by replacing  $a$  with  $aq$ ,  $b$  with  $bq^2$  and  $c$  with  $cq$  in  $w_{T'}(a, b, c; q)$ . Therefore, the generating function for all such  $T$  is given by

$$(b+c)(a+b)q \sum_{j=0}^{\infty} \sum_{T'} \sum_{k=0}^{\infty} a^j w_{T'}(aq, bq^2, cq) c^k q^k = \frac{(b+c)(a+b)q}{(1-a)(1-cq)} G(aq, bq^2, cq; q).$$

The factor of  $(b+c)$  accounts for the possibility of  $t_1$  being a gray or white square, the factor of  $(a+b)$  accounts for the possibility of  $t_2$  being a black or gray square, and the factor of  $q$  accounts for the fact that  $t_1$  is to the left of  $t_2$ . Adding the above two cases completes the proof.  $\square$

## 4 Specializations

In section 6 of [2], Fine considered several applications of iterations of the substitution  $t \rightarrow tq$ . In particular, Fine was led to identities involving  $F(a, 0; t; q)$  and  $F(a, 1; t; q)$ . We present here the corresponding equations for  $G(a, b, 0; q)$  and  $G(a, b, 1; q)$ .

In the case  $c = 0$  (i.e., the tiling has no white squares) we can count tilings based on the number of gray squares. More specifically, consider all tilings that have exactly  $n$  gray squares. Each such tiling starts with  $j \geq 0$  consecutive black squares followed by  $n$  segments, where each segment consists of a single gray square followed by any number of consecutive black squares.



If the  $i$ th segment contains  $k \geq 0$  black squares, then the weight of the  $i$ th gray square is  $bq^{i-1}$  and the weight of each of the black squares in the  $i$ th segment is  $aq^i$ . Allowing for all possible values of  $k$ , the weight of the  $i$ th segment contributes

$$bq^{i-1} \sum_{k=0}^{\infty} a^k q^{ik} = \frac{bq^{i-1}}{1-aq^i}$$

to the generating function  $G(a, b, 0; q)$ . Therefore, the generating function for tilings with no white squares and exactly  $n$  gray squares is given by

$$\begin{aligned} \sum_{j=0}^{\infty} a^j \prod_{i=1}^n \frac{bq^{i-1}}{1-aq^i} &= \frac{b^n q^{(n^2-n)/2}}{(1-a)(aq)_n} \\ &= \frac{b^n q^{(n^2-n)/2}}{(a)_{n+1}} \end{aligned}$$

and thus

$$\sum_{n=0}^{\infty} (-b/a)_n a^n = G(a, b, 0; q) = \sum_{n=0}^{\infty} \frac{b^n q^{(n^2-n)/2}}{(a)_{n+1}}.$$

In the case  $c = 1$ , Theorem 2 yields

$$G(a, b, 1; q) = \frac{1+b}{1-a} G(aq, bq, 1; q).$$

Applying this functional equation  $n$  times yields

$$\begin{aligned} G(a, b, 1; q) &= \frac{(1+b)(1+bq)(1+bq^2)}{(1-a)(1-aq)(1-aq^2)} \cdots \frac{(1+bq^{n-1})}{(1-aq^{n-1})} G(aq^n, bq^n, 1; q) \\ &= \frac{(-b)_n}{(a)_n} G(aq^n, bq^n, 1; q). \end{aligned}$$

If we assume that  $|q| < 1$ , taking the limit as  $n$  tends to infinity produces

$$\begin{aligned} G(a, b, 1; q) &= \frac{(-b)_{\infty}}{(a)_{\infty}} G(0, 0, 1; q) \\ &= \frac{(-b)_{\infty}}{(a)_{\infty}} \end{aligned}$$

since  $G(0, 0, 1; q) = 1$ . This completes a proof of Cauchy's  $q$ -analog of the binomial theorem [1] since

$$\sum_{n=0}^{\infty} \frac{(-b/a)_n a^n}{(q)_n} = G(a, b, 1; q) = \prod_{i=0}^{\infty} \frac{1+bq^i}{1-aq^i}.$$

Cauchy's result may also be obtained directly by pointing out that every tiling can be broken up into segments consisting of any number of consecutive black squares followed by a single gray or white square. In other words, every gray or white square marks the end of a segment.



Note that this means we now have to allow for tilings to end with a white square. In particular, since the weight of a white square is 1, we can think of our tilings as being an infinitely long sequence of tiles with only a finite number of black and/or gray squares.

## 5 Symmetry of $(1 - a)G(a, b, c; q)$

The final result in section 6 of [2] is that  $(1 - t)F(a, b; t : q)$  is invariant after the substitution  $a \rightarrow at/b$ ,  $b \rightarrow t$ , and  $t \rightarrow b$ . In this final section, we prove the same result in terms of the function  $(1 - a)G(a, b, c; q)$ . To this end, recall that

$$(1 - a)G(a, b, c; q)$$

can be interpreted as the generating function for tilings that do not start with a black square and as usual, do not end with a white square. We will show that this series is symmetric in the variables  $a$  and  $c$ . In other words,

$$(1 - a)G(a, b, c; q) = (1 - c)G(c, b, a; q) \tag{2}$$

or equivalently,

$$G(a, b, c; q) = \frac{1 - c}{1 - a}G(c, b, a; q).$$

We begin by noting that  $(1 - c)G(c, b, a; q)$  can be interpreted as the generating function for tilings that do not start with a black square and do not end with a white square, where the weight of each tiling is computed using the new weight function,  $\tilde{w}(t)$ , defined by

$$\tilde{w}(t) = \begin{cases} cq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \text{ } \square \text{ or } \square \text{ to its left} \\ bq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \text{ } \blacksquare \text{ or } \square \text{ to its left} \\ a & \text{if } t \text{ is a } \square \end{cases}$$

and  $\tilde{w}_T(a, b, c; q) = \prod_{t \in T} \tilde{w}(t)$ . To prove (2), it suffices to describe an involution,  $\varphi$ , on the set of tilings such that

$$w_T(a, b, c; q) = \tilde{w}_{\varphi(T)}(a, b, c; q)$$

Given a tiling  $T$ ,  $\varphi(T)$  is obtained by reversing the order of the tiles in  $T$  and then replacing each black square with a white square and vice versa. For example, if  $T$  is given by



then  $\varphi(T)$  is



Clearly this operation is an involution on the set of tilings that do not start with a black square and do not end with a white square. Furthermore, the number of black squares in  $T$  is equal to the number of white squares in  $\varphi(T)$ , the number of gray squares in  $T$  is equal to the number of gray squares in  $\varphi(T)$ , and the number of white squares in  $T$  is equal to the number of black squares in  $\varphi(T)$ . This implies that the power of  $a$ ,  $b$ , and  $c$ , respectively, in  $w_T(a, b, c; q)$  is the same as the power of  $a$ ,  $b$  and  $c$  in  $\tilde{w}_{\varphi(T)}(a, b, c; q)$ .

It remains to show that  $w_T(a, b, c; q)$  and  $\tilde{w}_{\varphi(T)}(a, b, c; q)$  have the same power of  $q$ . However, this is a simple consequence of the following observations:

- $T$  has a  $\square$  in position  $i$  and a  $\blacksquare$  in position  $j$  if and only if  $\varphi(T)$  has a  $\square$  in position  $n + 1 - j$  and a  $\blacksquare$  in position  $n + 1 - i$ .
- $T$  has a  $\square$  in position  $i$  and a  $\blacksquare$  in position  $j$  if and only if  $\varphi(T)$  has a  $\square$  in position  $n + 1 - j$  and a  $\square$  in position  $n + 1 - i$ .
- $T$  has a  $\square$  in position  $i$  and a  $\square$  in position  $j$  if and only if  $\varphi(T)$  has a  $\square$  in position  $n + 1 - j$  and a  $\blacksquare$  in position  $n + 1 - i$ .
- $T$  has a  $\square$  in position  $i$  and a  $\square$  in position  $j$  if and only if  $\varphi(T)$  has a  $\square$  in position  $n + 1 - j$  and a  $\square$  in position  $n + 1 - i$ .

where  $n$  is the total number of tiles in  $T$ . In other words, every factor of  $q$  that arises from the weight of a black or gray square in  $T$ , corresponds to a factor of  $q$  from the weight of a black or gray square in  $\varphi(T)$ , and vice versa. For example, the weight of  $\varphi(T)$  is given by

$$\begin{aligned} \tilde{w}_{\varphi(T)}(a, b, c; q) &= a \cdot cq \cdot bq \cdot cq^2 \cdot cq^2 \cdot a \cdot bq^3 \cdot cq^4 \cdot cq^4 \cdot cq^4 \cdot bq^4 \cdot cq^5 \cdot a \cdot cq^6 \\ &= a^3 b^3 c^8 q^{36} \\ &= w_T(a, b, c; q). \end{aligned}$$

## References

- [1] George E. Andrews. *The theory of partitions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.
- [2] Nathan J. Fine. *Basic hypergeometric series and applications*, volume 27 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988. With a foreword by George E. Andrews.
- [3] L. J. Rogers. On two theorems of combinatory analysis and some allied identities. *Proc. London Math. Soc.*, 16(2):315–336, 1917.