

# Combinatorics of the Lascoux-Schützenberger Tree

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# Stanley Symmetric Functions

Let  $Red(\sigma)$  denote the set of reduced words corresponding to  $\sigma$ . Define the Stanley symmetric function,  $F_\sigma(X)$ , as follows

$$F_\sigma(X) = \sum_{\substack{w \in Red(\sigma) \\ w = a_1 a_2 \cdots a_l}} \sum_{\substack{1 \leq b_1 \leq b_2 \leq \cdots \leq b_l \\ a_i > a_{i+1} \Rightarrow b_i < b_{i+1}}} x_{b_1} x_{b_2} \cdots x_{b_l}$$

**Theorem 1** *For any permutation  $\sigma$ ,  $F_\sigma(X)$  is Schur positive.*

# Lascoux-Schützenberger Tree

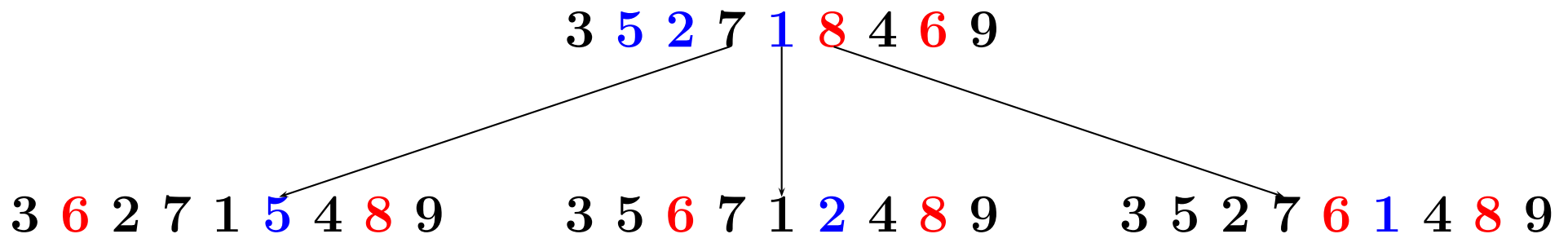
For a given permutation  $\sigma$ , define

$$\mathbf{r} = \max(i \mid \sigma_i > \sigma_{i+1}) = \text{“last descent”}$$

$$\mathbf{s} = \max(i > \mathbf{r} \mid \sigma_i < \sigma_{\mathbf{r}}),$$

$$\mathbf{I} = \{i < \mathbf{r} \mid l(\sigma t_{\mathbf{r},\mathbf{s}} t_{i,\mathbf{r}}) = l(\sigma t_{\mathbf{r},\mathbf{s}}) + 1\}$$

Example:





Define:

$$\Phi(\sigma) = \begin{cases} \{\sigma t_{\mathbf{r},s} t_{\mathbf{i},\mathbf{r}} \mid \mathbf{i} \in \mathbf{I}\} & \text{if } \mathbf{I} \neq \emptyset \\ \Phi(1 \otimes \sigma) & \text{otherwise} \end{cases}$$

where

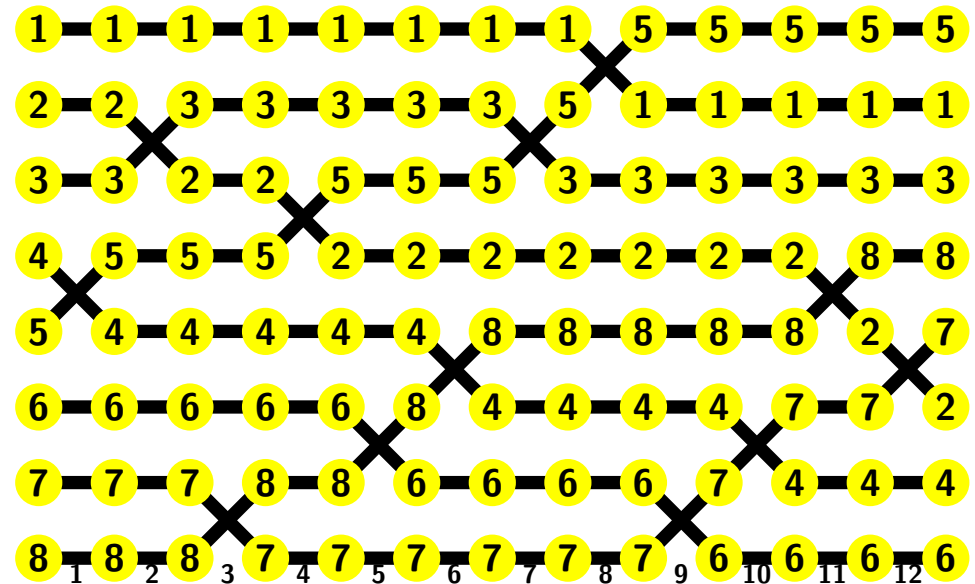
$$1 \otimes \sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n+1 \\ 1 & \sigma_1 + 1 & \sigma_2 + 1 & \dots & \sigma_n + 1 \end{pmatrix}$$

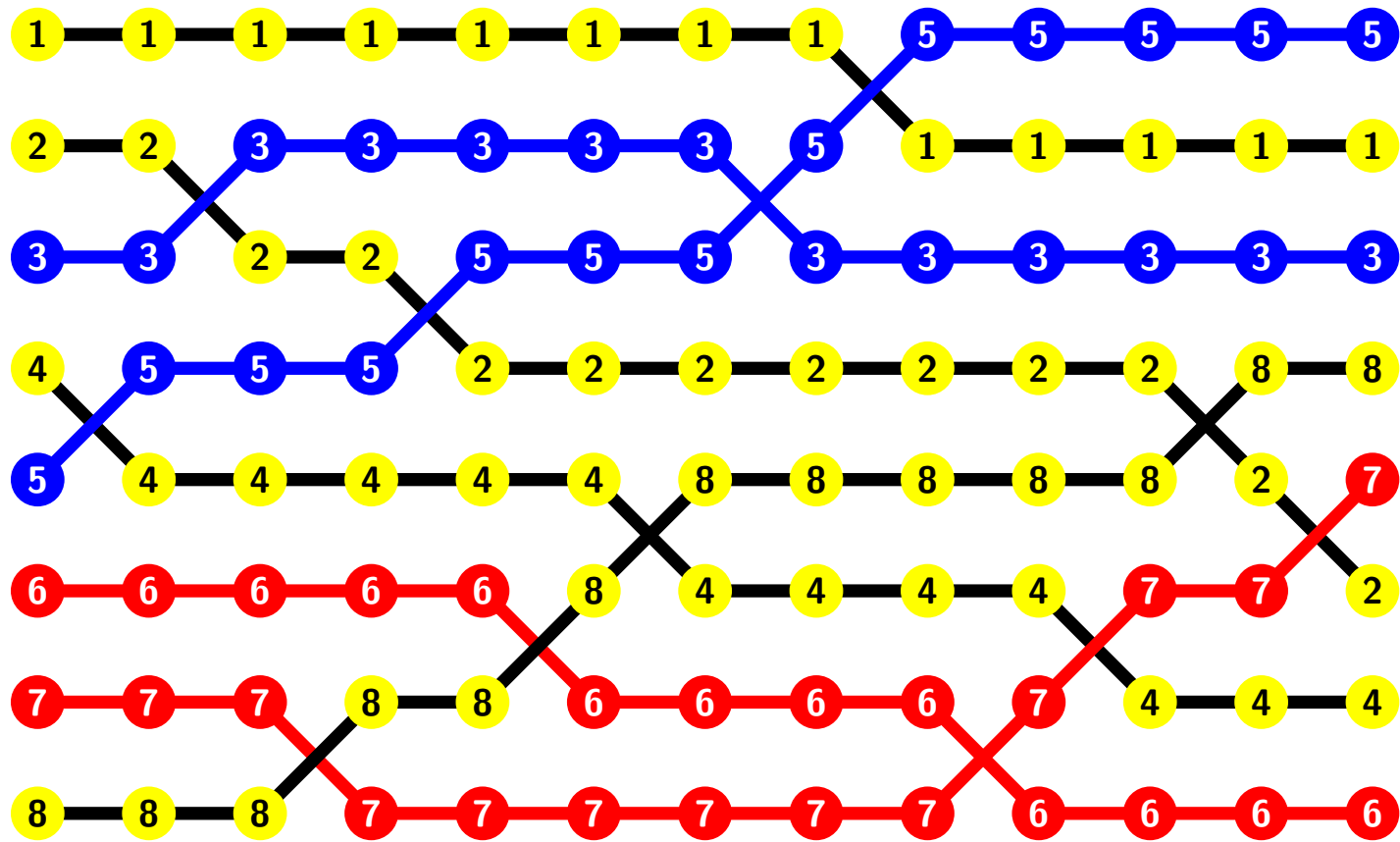
**Theorem 2** *For any permutation  $\sigma$ , we have*

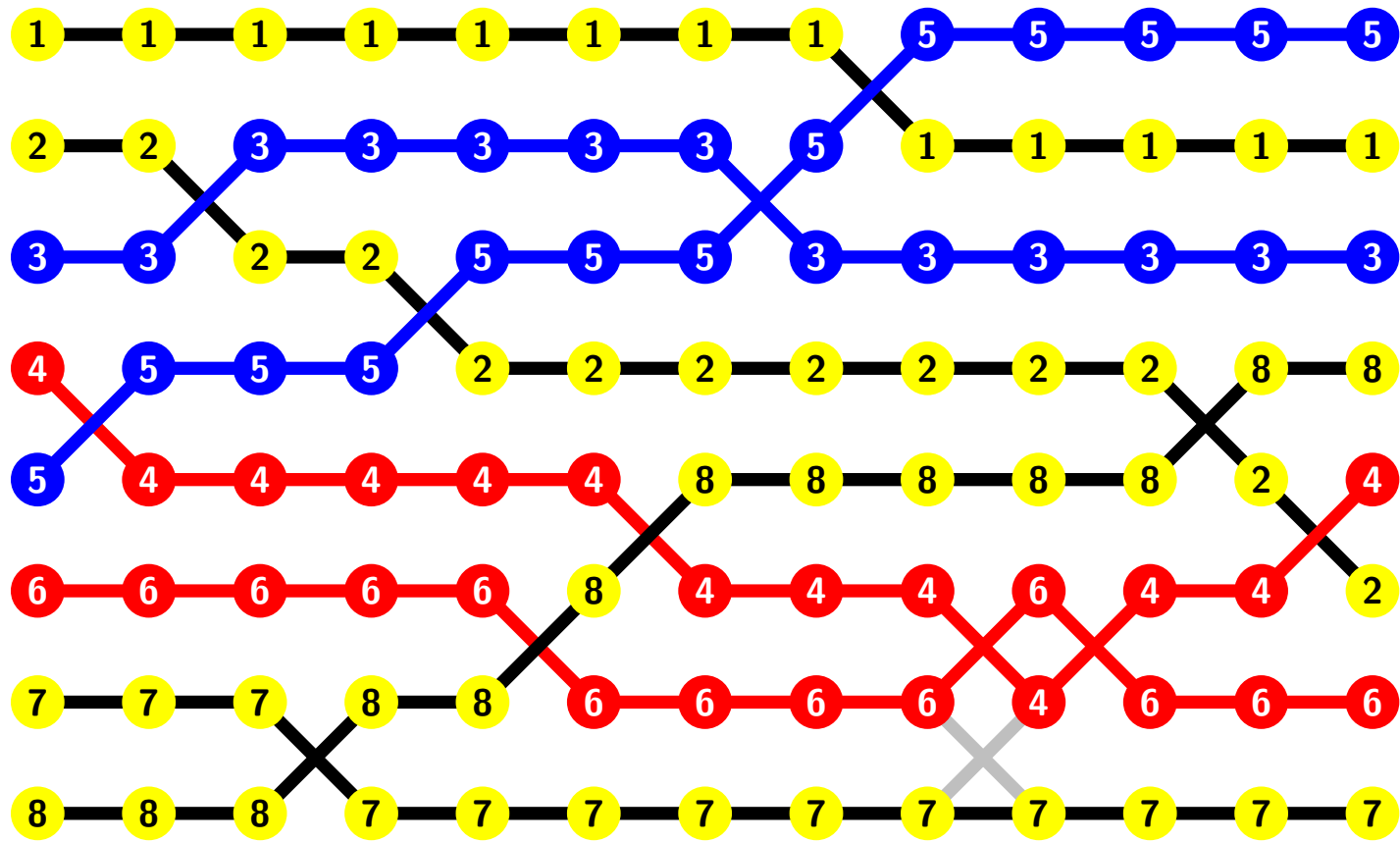
$$F_\sigma(X) = \sum_{\sigma' \in \Phi(\sigma)} F_{\sigma'}(X).$$

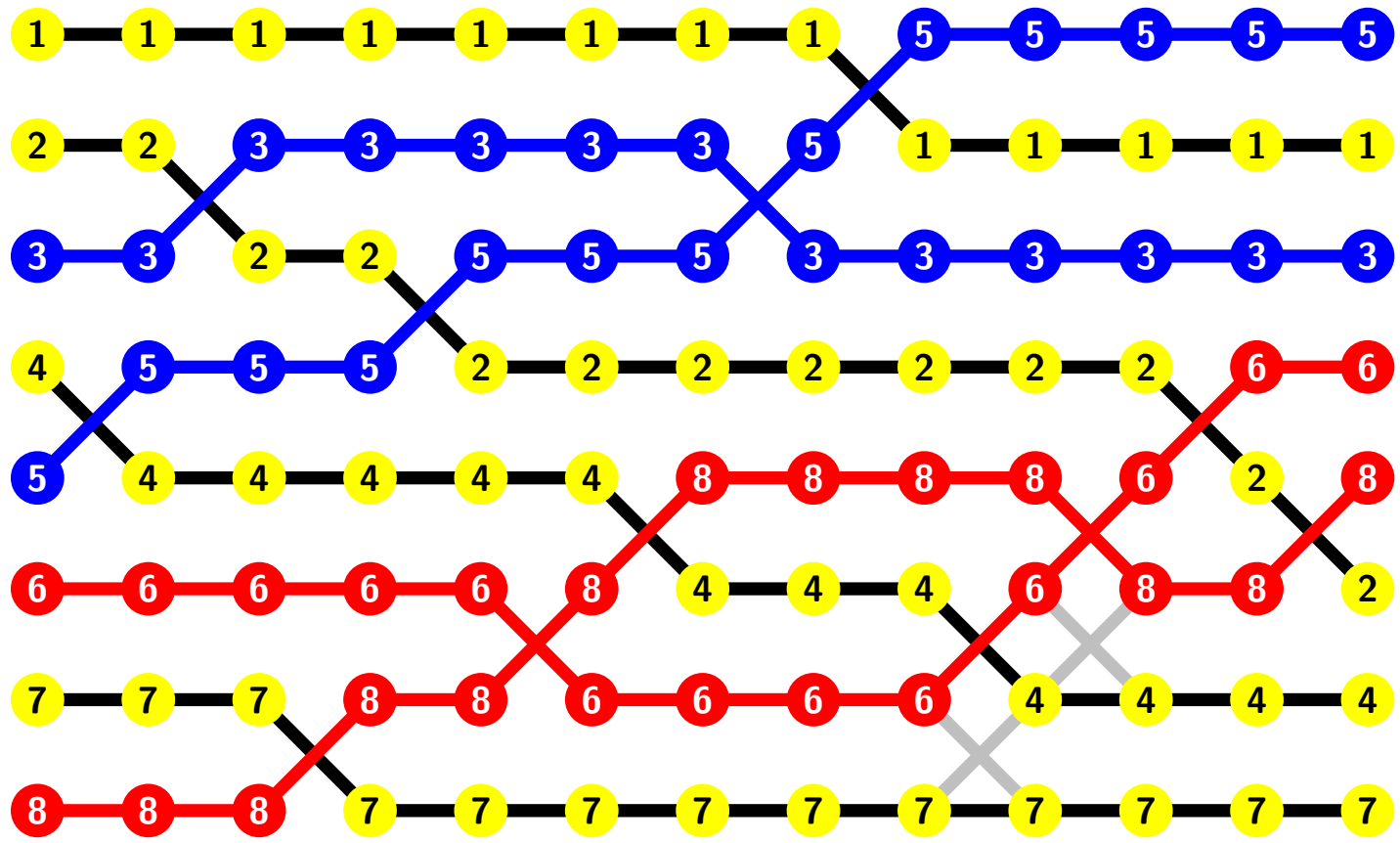
# Labeled Circle and Line Diagrams

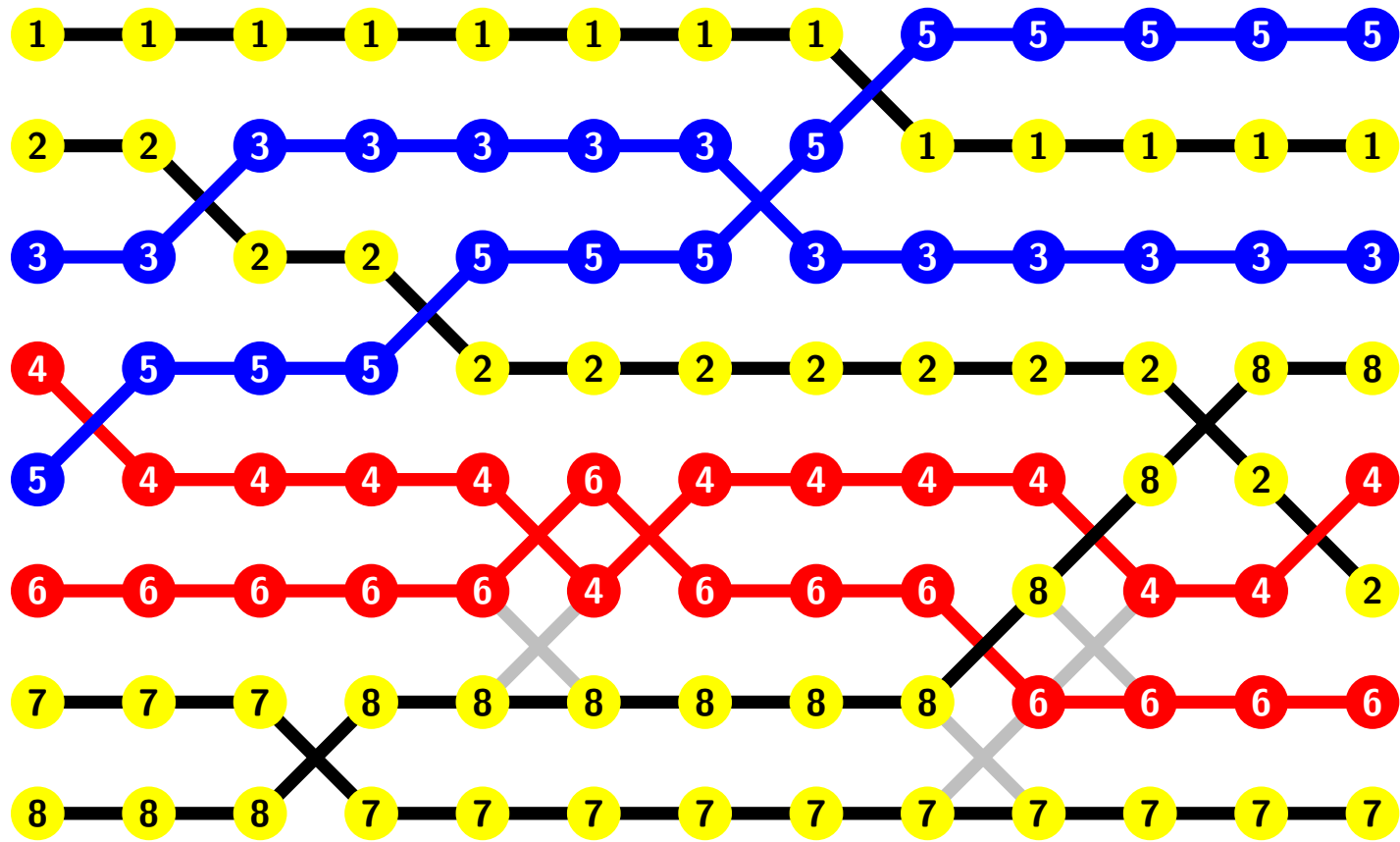
	5	1	3	8	7	2	4	6
1	8	×	•	•	•	•	•	•
2	4	•	2	11	12	×	•	•
3	7	•	×	•	•	•	•	•
4	1	•	•	6	10	•	×	•
5	×	•	•	•	•	•	•	•
6	•	•	•	5	9	•	•	×
7	•	•	•	3	×	•	•	•
8	•	•	•	×	•	•	•	•

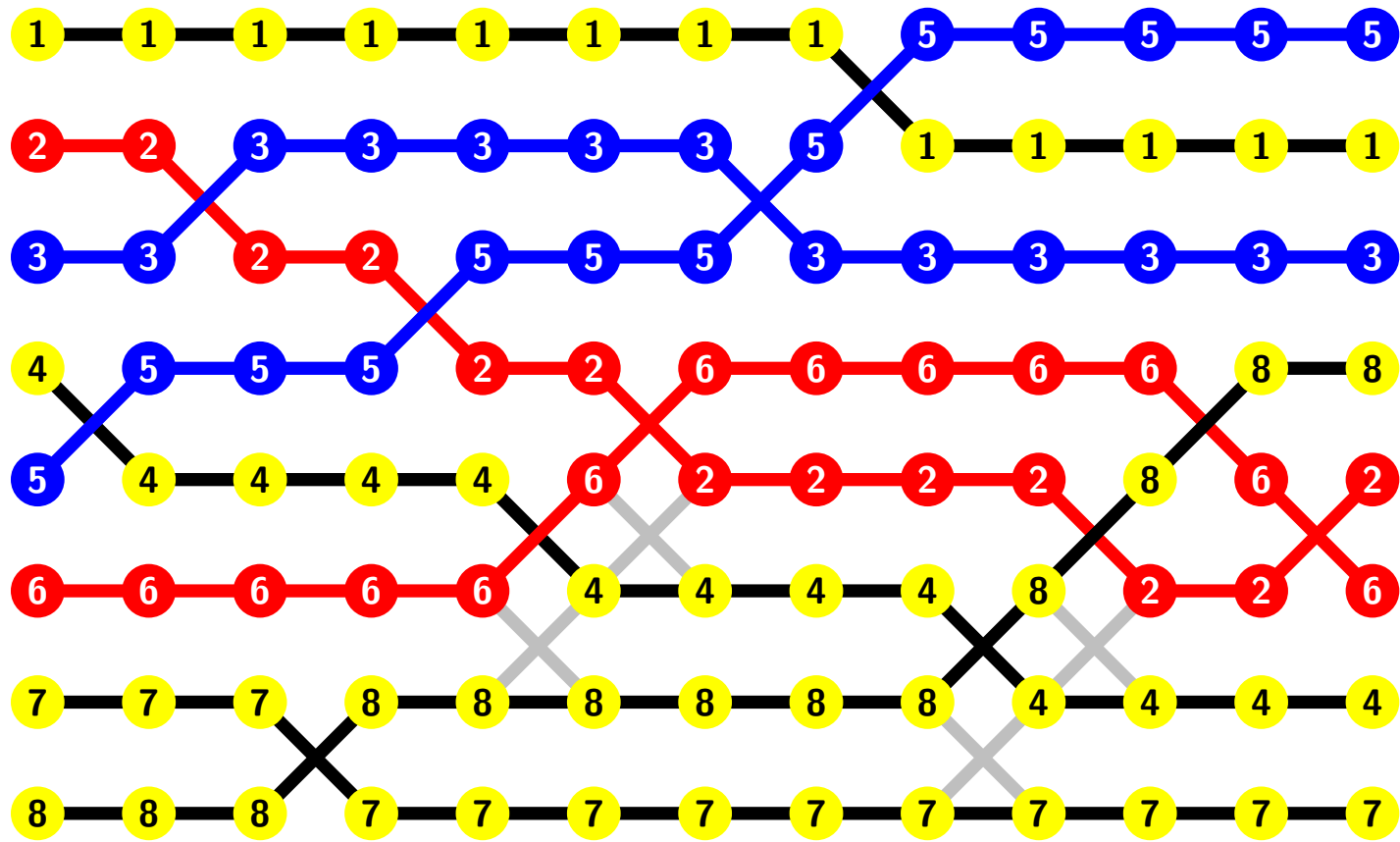


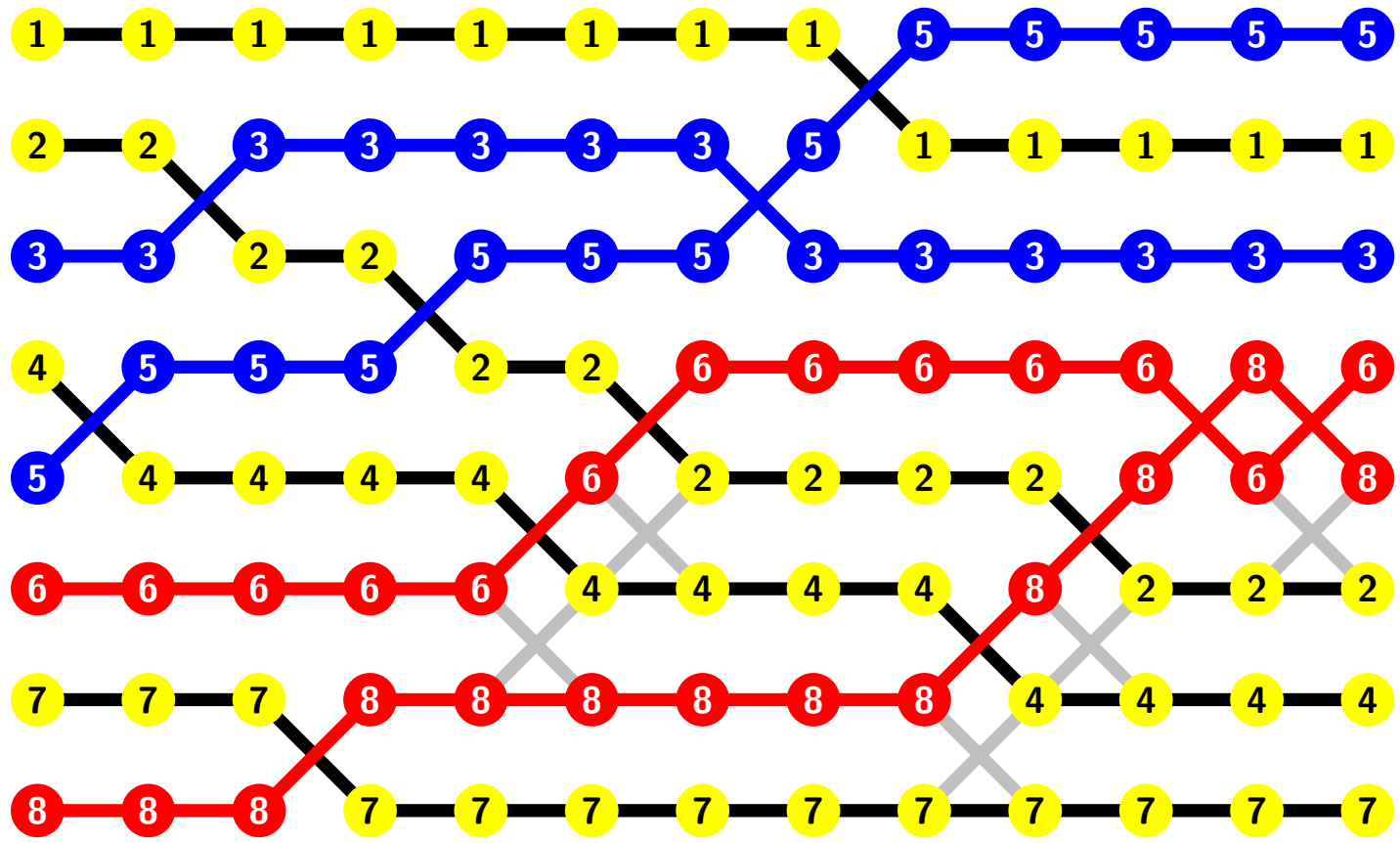


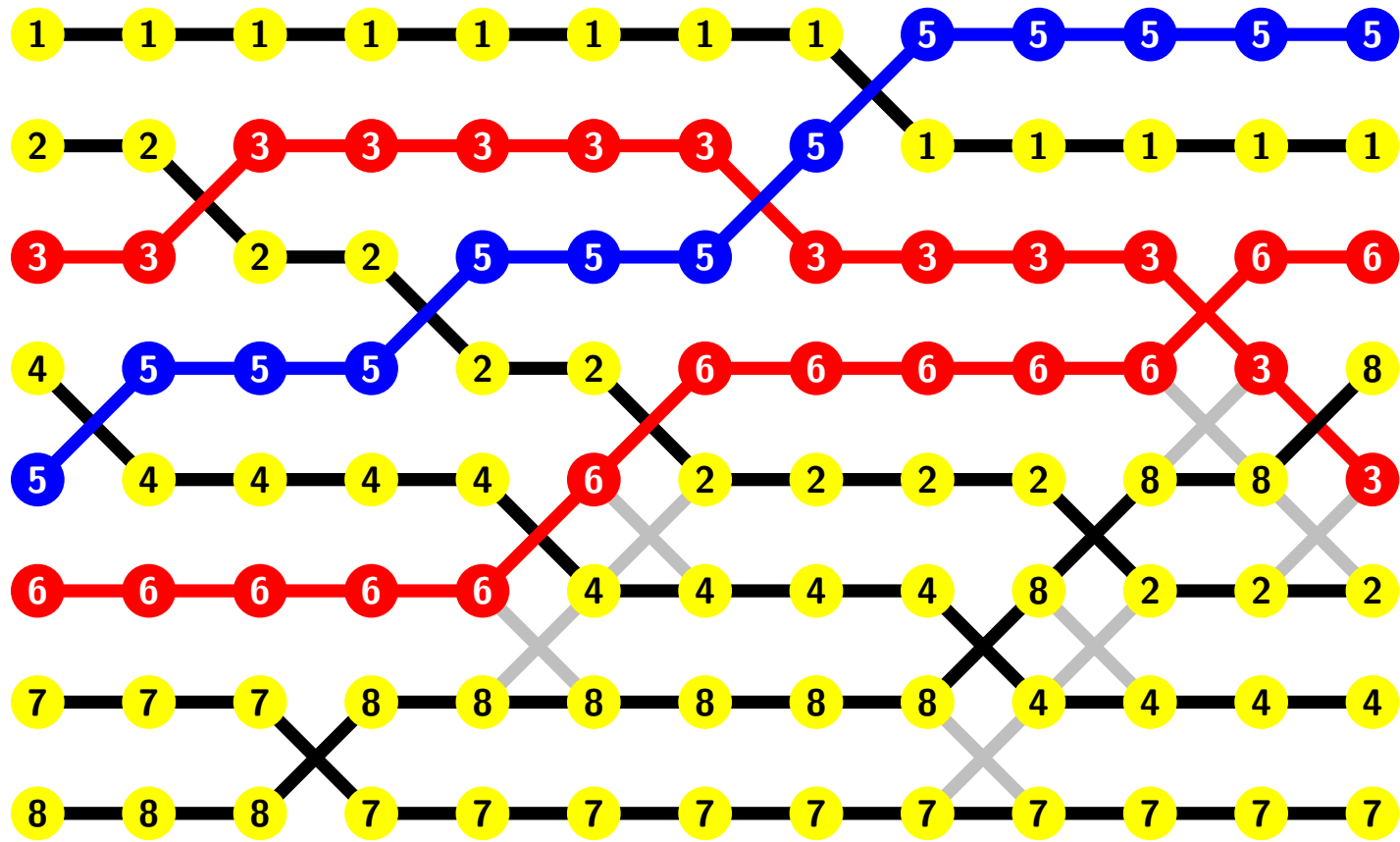










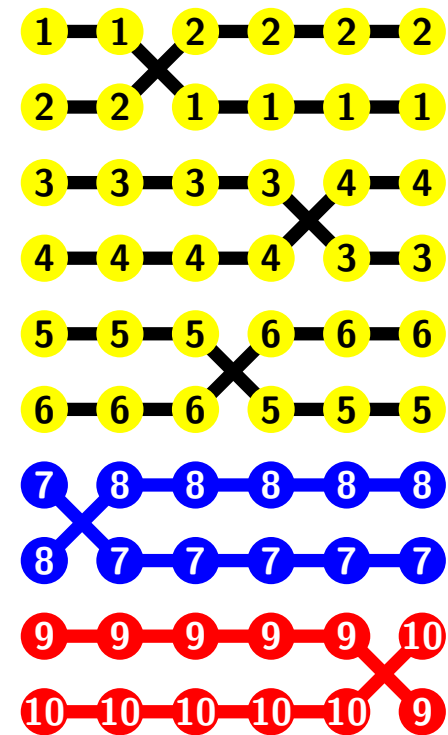


## Why it Works

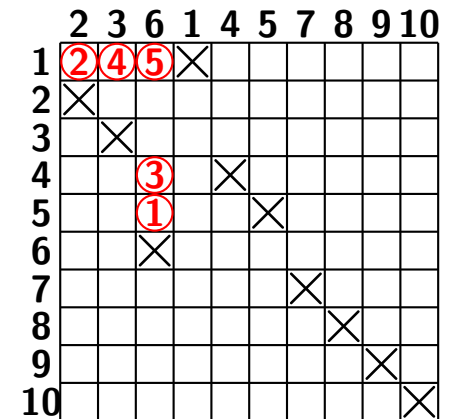
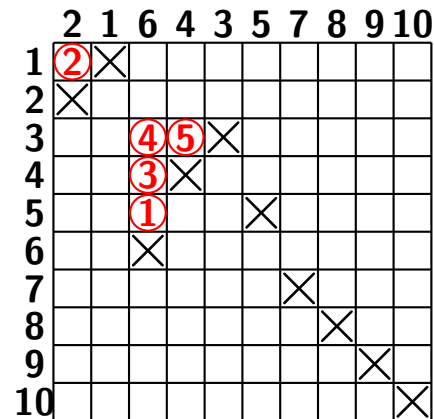
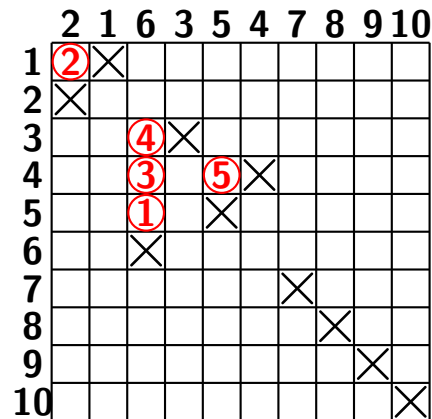
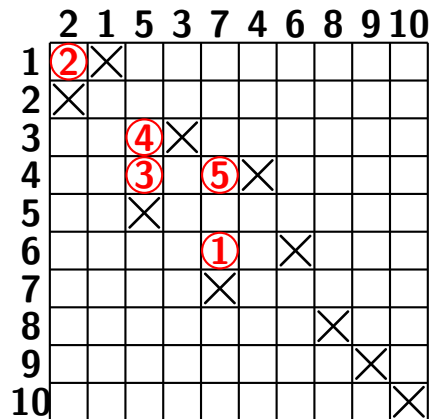
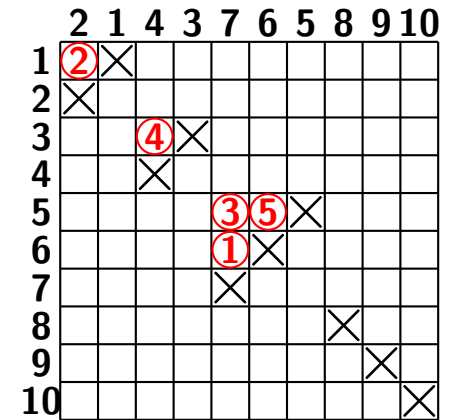
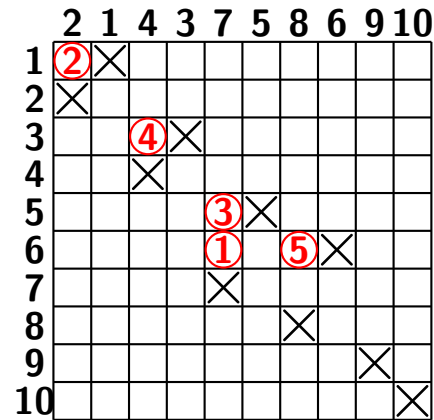
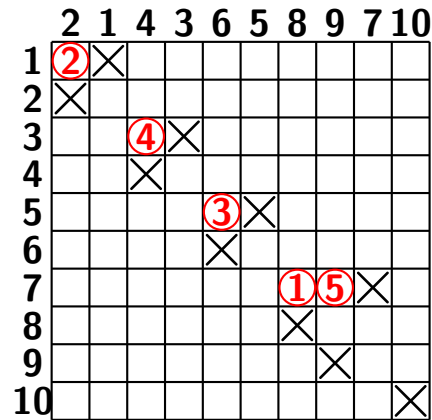
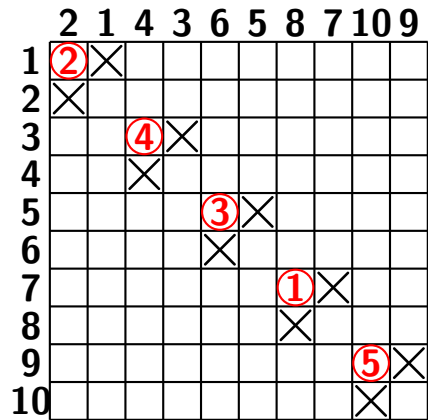
1. “Other” cross is unique.
2. Each letter is bumped at most one. This insures that process will terminate and that descent sets are preserved.
3. At any given step, word corresponds to the permutation  $\sigma_{t_{r,s}, t_{i,r}}$  for some  $i$ . Algorithm will stop when  $i \in I$ .
4. Inverse is defined using the same mechanism.

# Alternating Permutation

	2	1	4	3	6	5	8	7	10	9
1	2	×	•	•	•	•	•	•	•	•
2	×	•	•	•	•	•	•	•	•	•
3	•	•	4	×	•	•	•	•	•	•
4	•	•	×	•	•	•	•	•	•	•
5	•	•	•	•	3	×	•	•	•	•
6	•	•	•	•	×	•	•	•	•	•
7	•	•	•	•	•	•	1	×	•	•
8	•	•	•	•	•	•	×	•	•	•
9	•	•	•	•	•	•	•	•	5	×
10	•	•	•	•	•	•	•	•	×	•

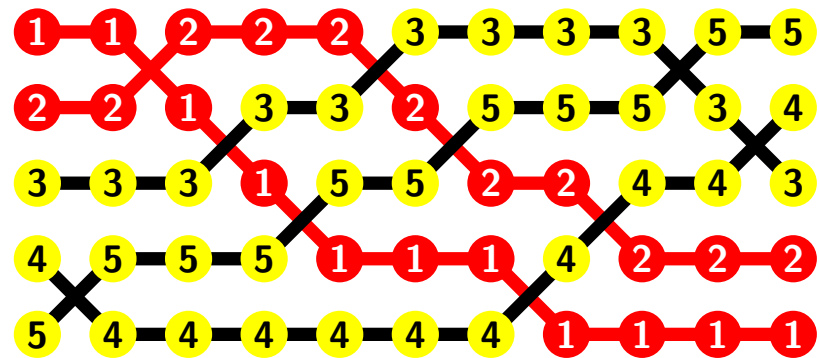


# Permutation To Standard Tableau

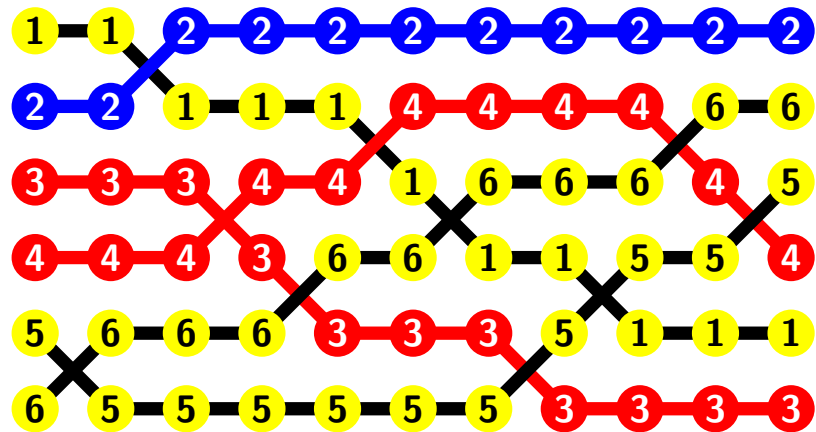


# Reverse Permutation

	5	4	3	2	1
1	4	7	3	2	×
2	6	8	5	×	
3	9	10	×		
4	1	×			
5	×				



	2	6	5	4	1	3
1	2	6	8	5	×	
2	×					
3		4	7	3		×
4		9	10	×		
5		1	×			
6		×				



# Balanced To Standard Tableau

	5	4	3	2	1
1	4	7	3	2	×
2	6	8	5	×	
3	9	10	×		
4	1	×			
5	×				

	2	6	5	4	1	3
1	2	6	8	5	×	
2	×					
3		4	7	3		×
4		9	10	×		
5		1	×			
6	×					

	3	6	5	2	1	4
1	3	6	8	2	×	
2	5	9	10	×		
3	×					
4		4	7			×
5		1	×			
6	×					

	2	4	7	6	1	3	5
1	2	5	9	10	×		
2	×						
3		3	6	8		×	
4		×					
5			4	7			×
6			1	×			
7			×				

	2	5	7	4	1	3	6
1	2	6	8	5	×		
2	×						
3		4	7	3		×	
4		9	10	×			
5		×					
6			1				×
7			×				

	3	5	7	2	1	4	6
1	3	6	8	2	×		
2	5	9	10	×			
3	×						
4		4	7			×	
5		×					
6			1				×
7			×				

	2	4	6	8	1	3	5	7
1	2	5	9	10	×			
2	×							
3		3	6	8		×		
4		×						
5			4	7			×	
6			×					
7				1				×
8				×				

## Conclusions

- Combinatorial proof of the Schur positivity of  $F_\sigma(X)$ .
- Combinatorial proof of Littlewood-Richardson rule.

## Conjectures

- Robinson-Schensted correspondence is a particular case when  $\sigma = (2, 1, 4, 3, \dots, 2n, 2n - 1)$ .
- Edelman-Greene bijection is a particular case when  $\sigma = (n, n - 1, \dots, 2, 1)$ .
- Jeu de taquin property.