The Littlewood-Richardson Rule
via
Reduced Factorizations of the Symmetric Group

David Little
Dartmouth College
September 25, 2003

http://www.math.dartmouth.edu/˜dlittle

Abstract

The collection of Schur functions is a standard basis for the ring of symmetric functions. The Littlewood-Richardson rule is a method used to expand the product of two Schur functions in terms of this standard basis. From a representation theoretical point of view, this is equivalent to finding the multiplicty of an irreducible representation in the tensor product of two irreducible representations of the symmetric group.

In this talk, we will show how a recent discovery in the realm of reduced factorizations of the symmetric group has lead to a new combinatorial proof of the Littlewood-Richardson rule. Time permitting, we will demonstrate how this discovery relates to the Robinson-Schensted and Edelman-Greene correspondences. No prior knowledge of symmetric functions and/or reduced factorizations of the symmetric group will be assumed.

Ring of Symmetric Functions

Let $f \in \mathbb{Q}[x_1, x_2, \ldots, x_N]$. f is said to be *symmetric* if for every $\sigma \in S_N$

$$f(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_N}) = f(x_1, x_2, \ldots, x_N).$$

Clearly we have that if f and g are symmetric, then

$$f + g \quad \text{and} \quad f \times g$$

are also symmetric and it follows that the collection of symmetric functions has the structure of a ring. We will denote the ring of symmetric functions in the variables $X = \{x_1, x_2, \ldots, x_N\}$ by $\Lambda(X)$.

Standard Bases of $\Lambda(X)$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ be a weakly decreasing sequence of nonnegative integers.

1. Monomial symmetric functions, m_λ

$$m_\lambda(X) = \sum_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}$$

2. Homogeneous symmetric functions, h_λ

$$h_k(X) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le N} x_{i_1} x_{i_2} \cdots x_{i_k} \qquad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_N}$$

3. Elementary symmetric functions, e_λ

$$e_k(X) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le N} x_{i_1} x_{i_2} \cdots x_{i_k} \qquad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_N}$$

4. Power symmetric functions, p_λ

$$p_k(X) = \sum_{i=1}^{N} x_i^k \qquad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_N}$$

Schur Functions

- Column Strict Tableaux of shape λ, $CST(\lambda)$, are labelings of the cells of the Ferrers diagram of λ with the numbers 1 through N that are weakly increasing from left to right and strictly increasing from bottom to top.

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} 5 \\ \cline{1-2} 3 & 4 \\ \hline 2 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 3 \\ \hline \end{array} \qquad w(T) = x_1^3 x_2^2 x_3^3 x_4^2 x_5$$

- $S_\lambda(X) = \sum_{T \in CST(\lambda)} w(T)$
- The collection of all Schur functions forms a basis for $\Lambda(X)$.

Problem

Given two partitions $\lambda \vdash n$ and $\mu \vdash m$, determine the coefficients $c^{\nu}_{\lambda,\mu}$ such that

$$S_{\lambda}(X)S_{\mu}(X) = \sum_{\nu \vdash n+m} c^{\nu}_{\lambda,\mu} S_{\nu}(X).$$

The coefficients $c^{\nu}_{\lambda,\mu}$ are known as Littlewood-Richardson coefficients and have the following representation theoretical interpretation:

$$A^{\lambda} \times A^{\mu} \uparrow^{S_{n+m}}_{S_n \times S_m} = \bigoplus_{\nu \vdash n+m} (A^{\nu})^{\oplus c^{\nu}_{\lambda,\mu}}$$

where A^{λ}, A^{μ} and A^{ν} are irreducible representations of S_n, S_m and S_{n+m}, respectively. And therefore

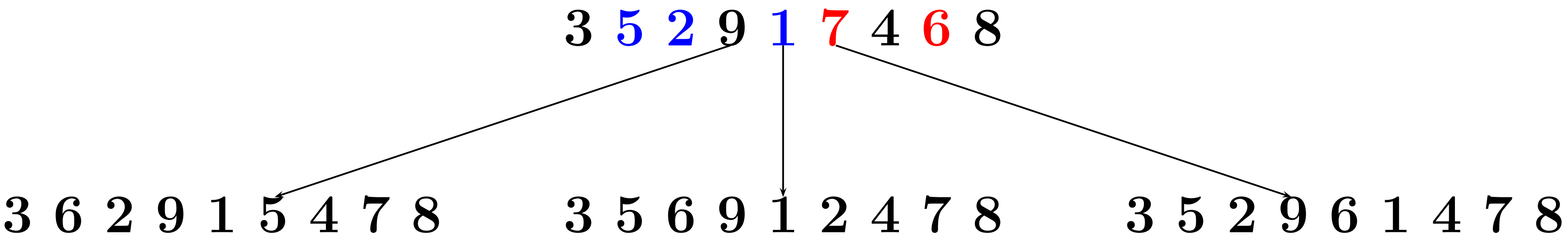
$$\chi^{A^{\lambda} \times A^{\mu} \uparrow} = \sum_{\nu \vdash n+m} c^{\nu}_{\lambda,\mu} \chi^{A^{\nu}}.$$

Children of a Permutation

For a given permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$, define

$$\begin{array}{rcl}
\mathbf{r} & = & \max(i \mid \sigma_i > \sigma_{i+1}) = \text{“last descent”} \\
\mathbf{s} & = & \max(i > \mathbf{r} \mid \sigma_i < \sigma_{\mathbf{r}}), \\
\mathbf{I} & = & \{i < \mathbf{r} \mid \sigma_i < \sigma_{\mathbf{s}} \ \ \& \ \ \forall j \in (i, \mathbf{r}) \quad \sigma_j \notin (\sigma_i, \sigma_{\mathbf{s}})\} \\
Children(\sigma) & = & \{\sigma(\mathbf{r}, \mathbf{s})(\mathbf{i}, \mathbf{r}) \mid \mathbf{i} \in \mathbf{I}\}
\end{array}$$

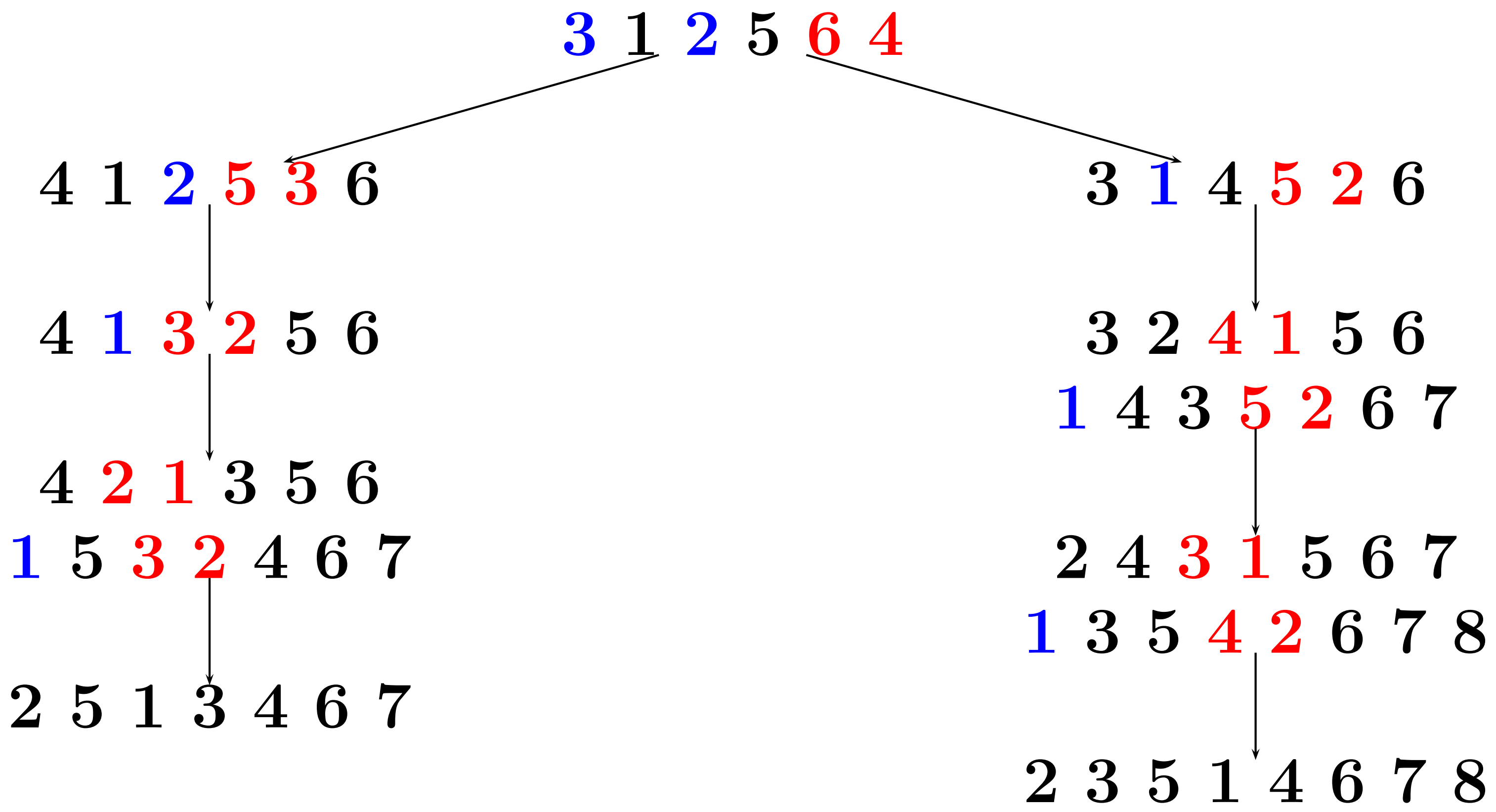
Example:



3 5 2 9 1 7 4 6 8

3 6 2 9 1 5 4 7 8 **3 5 6 9 1 2 4 7 8** **3 5 2 9 6 1 4 7 8**

Lascoux-Schützenberger Tree



3 1 2 5 6 4

4 1 2 5 3 6

4 1 3 2 5 6

4 2 1 3 5 6
1 5 3 2 4 6 7

2 5 1 3 4 6 7

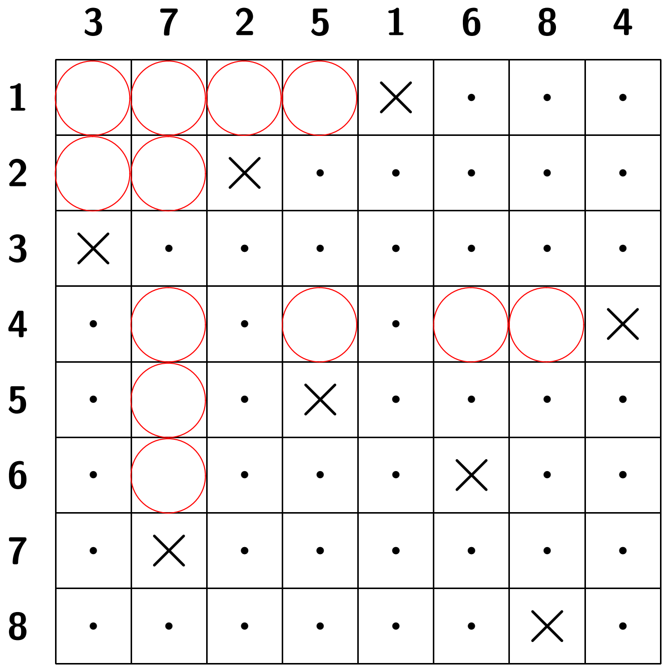
3 1 4 5 2 6

3 2 4 1 5 6
1 4 3 5 2 6 7

2 4 3 1 5 6 7
1 3 5 4 2 6 7 8

2 3 5 1 4 6 7 8

The Shape of a Permutation



	3	7	2	5	1	6	8	4
1	○	○	○	○	×	·	·	·
2	○	○	×	·	·	·	·	·
3	×	·	·	·	·	·	·	·
4	·	○	·	○	·	○	○	×
5	·	○	·	×	·	·	·	·
6	·	○	·	·	·	×	·	·
7	·	×	·	·	·	·	·	·
8	·	·	·	·	·	·	×	·

The code of σ is the vector whose i^{th} component consists of the number of circles in the i^{th} column of it's diagram.

The shape of σ, denoted $\lambda(\sigma)$ is the decreasing rearrangement of it's code.

$$\lambda(\sigma) = (5, 2, 2, 1, 1, 1)$$

Littlewood-Richardson Rule (Lascoux-Schützenberger)

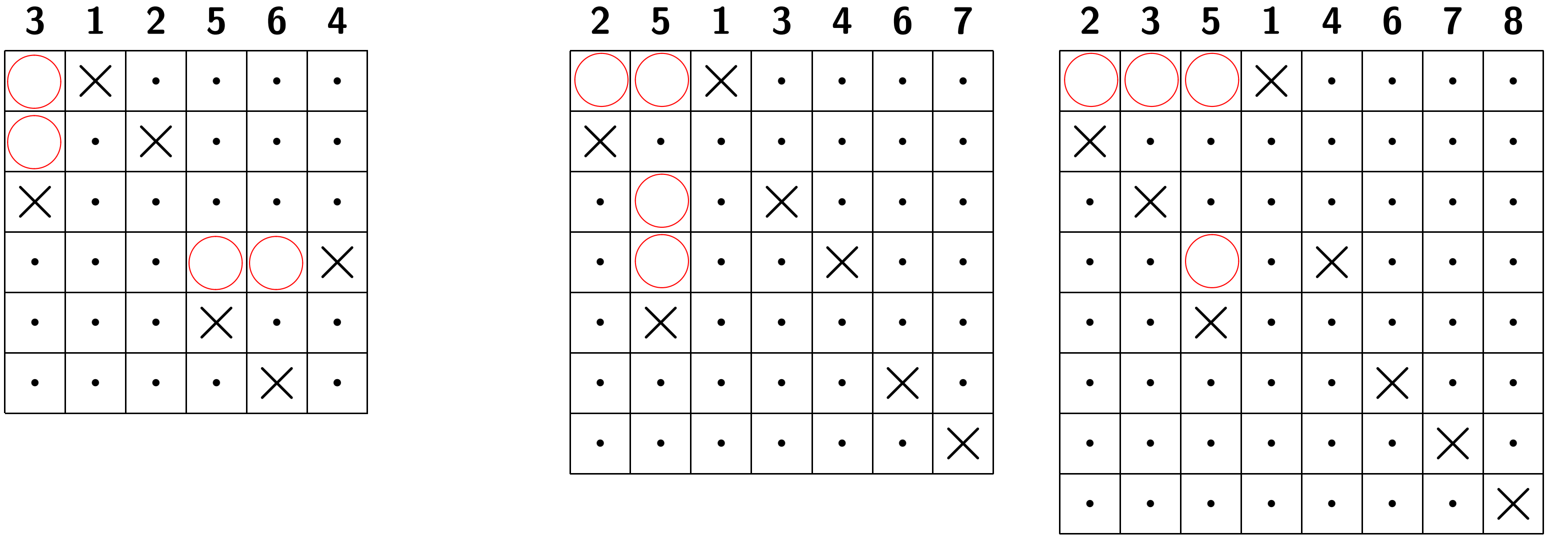
Let $\alpha \in S_n$ and $\beta \in S_m$ be *Grassmanian* permutations, that is, α and β each have exactly one descent. Then the product $S_\lambda(X) S_\mu(X)$ where $\lambda = \lambda(\alpha)'$ and $\mu = \lambda(\beta)'$ can be expanded in the following manner:

1. Form the Lascoux-Schützenberger Tree corresponding to $\alpha \otimes \beta$ where
$$\alpha \otimes \beta = (\alpha_1, \ldots, \alpha_n, n + \beta_1, \ldots, n + \beta_m)$$
2. Let $Leaves = \{\sigma\}$ where σ is a leaf in the L-S Tree.
3. Then
$$S_\lambda(X) S_\mu(X) = \sum_{\gamma \in Leaves} S_{\lambda(\gamma)'}(X)$$
and thus
$$c^\nu_{\lambda,\mu} = |\{\gamma \in Leaves \mid \lambda(\gamma)' = \nu\}|.$$

Example

Let $\alpha = (3, 1, 2)$ and $\beta = (2, 3, 1)$. Therefore

$$\alpha \otimes \beta = (3, 1, 2, 5, 6, 4).$$



3	1	2	5	6	4
○	×	·	·	·	·
○	·	×	·	·	·
×	·	·	·	·	·
·	·	·	○	○	×
·	·	·	×	·	·
·	·	·	·	×	·

2	5	1	3	4	6	7
○	○	×	·	·	·	·
×	·	·	·	·	·	·
·	○	·	×	·	·	·
·	○	·	·	×	·	·
·	×	·	·	·	·	·
·	·	·	·	·	×	·
·	·	·	·	·	·	×

2	3	5	1	4	6	7	8
○	○	○	×	·	·	·	·
×	·	·	·	·	·	·	·
·	×	·	·	·	·	·	·
·	·	○	·	×	·	·	·
·	·	×	·	·	·	·	·
·	·	·	·	·	×	·	·
·	·	·	·	·	·	×	·
·	·	·	·	·	·	·	×

$$S_{1,1}(X)S_2(X) = S_{2,1,1}(X) + S_{3,1}(X)$$

Reduced Factorizations of the Symmetric Group

The symmetric group, S_N, is generated by simple transpositions $s_i = (i, i+1)$ for $1 \leq i < N$ These generators satisfy the Coxeter relations

$$\begin{aligned} &a)\ s_i^2 = id \ \text{ for } 1 \leq i \leq n-1, \\ &b)\ s_i s_j = s_j s_i \ \text{ for } |i-j| \geq 2, \\ &c)\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \ \text{ for } 1 \leq i < N-1. \end{aligned}$$

A *word* $w = a_1 a_2 \cdots a_l \in [N-1]^*$ corresponds to $\sigma \in S_N$ if

$$\sigma = s_{a_1} s_{a_2} \cdots s_{a_l}$$

w is said to be *reduced* if $l = l(\sigma) := |\{(\sigma_i, \sigma_j) \mid i < j \ \ \sigma_i > \sigma_j\}|$

Stanley Symmetric Functions

Let $Red(\sigma)$ denote the set of reduced words corresponding to σ. Define the Stanley symmetric function, $F_\sigma(X)$, as follows

$$F_\sigma(X) = \sum_{\substack{w \in Red(\sigma) \\ w = a_1 a_2 \cdots a_l}} \sum_{\substack{1 \leq b_1 \leq b_2 \leq \cdots \leq b_l \leq N \\ a_i > a_{i+1} \Rightarrow b_i < b_{i+1}}} x_{b_1} x_{b_2} \cdots x_{b_l}$$

Theorem 1 *For any permutation σ, $F_\sigma(X)$ is Schur positive.*

Properties of $F_\sigma(X)$

Let $\alpha \in S_n$ and $\beta \in S_m$.

1. $F_{\alpha \otimes \beta}(X) = F_\alpha(X) \times F_\beta(X)$
2. $F_\alpha(X) = S_{\lambda(\alpha)'}(X)$ if α is Grassmanian.
3. $F_\alpha(X) = \sum_{\sigma \in Leaves(\alpha)} F_{\lambda(\sigma)}(X)$

In the case when α and β are Grassmanian, the above properties imply

$$S_{\lambda(\alpha)'}(X) S_{\lambda(\beta)'}(X) = \sum_{\sigma \in Leaves(\alpha \otimes \beta)} S_{\lambda(\sigma)'}(X)$$

Proof of Littlewood-Richardson Rule

Theorem 2

$$F_\alpha(X) = \sum_{\sigma \in Children(\alpha)} F_\sigma(X)$$

Proof: We will demonstrate a bijection, θ, between the following two collections of reduced words

$$Red(\alpha) \quad \text{and} \quad \bigcup_{\sigma \in Children(\alpha)} Red(\sigma)$$

such that if $a = a_1 a_2 \cdots a_l \in Red(\alpha)$ and $\theta(a) = w_1 w_2 \cdots w_l \in Red(\sigma)$ for some σ in $Children(\alpha)$ then

$$a_i > a_{i+1} \Leftrightarrow w_i > w_{i+1}.$$