Group Project: Combinatorics and Games of Chance

1 Introduction

There is a very strong connection between the methods of counting that we have studied thus far and the foundations of probability theory. Games of chance, be it flipping a coin, picking numbers out of a hat, or your favorite card/dice game, can all be analyzed from a purely combinatorial point of view. The goal of this project is to specifically analyze the dice game of Yahtzee.

2 Rules and Regulations

Students will work in groups consisting of no more than 4 people. Each group will submit one typed report, including a cover page with project title and names of group members. Your report should have an introduction, giving a brief description of the project and any other information or basic definitions/assumptions you will be using that are relevant to the entire project. Each of the three parts of the project should also include a brief introduction describing the specific experiment and any other information that is relevant to that part of the project.

The group report will be graded out of 75 points (15% of your overall grade). Each project consists of three parts (see below). Part I is worth 20 points, Part II is worth 20 points and Part III is worth 20 points. Each of these parts will be graded for mathematical detail and accuracy. The remaining 15 points will be awarded based on presentation (clarity and consistency of writing style, grammatical correctness, appropriate use of notation, terminology, and examples, etc.)

Make sure to tell me who is in your group and which project you will be working on by **Friday, March 27**. During the week of **April 13 through April 17** or before, your group must show me a rough draft of your report, at which point I will make any necessary comments and/or suggestions. **The deadline to submit your completed project is Monday, April 27.**

3 Background on Probability

Suppose we conduct an experiment that has n distinct outcomes where each outcome is equally likely. That is, each outcome occurs with probability $1/n$. For example, selecting 5 cards from a deck of 52, rolling 5 dice or selecting 5 numbers between 1 and 55 would all qualify as experiments where each outcome (for each individual experiment) is equally likely. Let S be the collection of all possible outcomes from our experiment. In probability theory, S is referred to as the *sample space*.

Any combination (i.e., subset) of S is referred to as an *event*. For example, selecting 5 cards all of the same suit would be an event. If an event A consists of r distinct outcomes from our experiment, then the probability of event A, denoted by $P(A)$, is given by

$$P(A) = \frac{\text{number of outcomes in the event } A}{\text{number of outcomes in the sample space } S} = \frac{r}{n}.$$

Consider the following experiment. Flip a coin 10 times and record an "H" if it lands on heads or a "T" if it lands on tails for each coin flip. Our sample space S consists of all sequences of H's and T's of length 10 (much like binary sequences except that we are using H's and T's instead of 0's and 1's). Let A be the event that we record exactly 8 H's in our experiment. Therefore

$$P(A) = \frac{\binom{10}{8}}{2^{10}} \approx 0.04395$$

In other words, this event occurs approximately 4.395% of the time.

The function $P(A)$ defined above satisfies properties very similar to the Sum and Product Rules that we have been using for counting. In particular, if the outcomes in event A can be partitioned into events $A_1, A_2, \ldots, A_k$ (i.e. $\{A_1, A_2, \ldots, A_k\}$ is a set partition of A) then

$$P(A) = P(A_1) + P(A_2) + \cdots + P(A_k).$$

On the other hand, if each outcome in A can be constructed using the product rule, that is to say that each event can be uniquely constructed from a series of experiments (A_i is an event of the ith experiment) and the number of outcomes in A_i is independent of previous outcomes, then

$$P(A) = P(A_1) \times P(A_2) \times \cdots \times P(A_k).$$

For example, the probability of flipping a coin 10 times and recording 7 or 8 heads would be given by

$$P(7 \text{ H's out of 10 coin flips}) + P(8 \text{ H's out of 10 coin flips}) = \frac{\binom{10}{7}}{2^{10}} + \frac{\binom{10}{8}}{2^{10}}.$$

The probability of flipping a coin 10 times and recording 4 heads and then flipping a coin 15 times and recording 9 heads would be given by

$$P(4 \text{ H's out of 10 coin flips}) \times P(9 \text{ H's out of 15 coin flips}) = \frac{\binom{10}{4}}{2^{10}} \times \frac{\binom{15}{9}}{2^{15}}.$$

One last example. Suppose that we flip 10 distinct coins and any coin that initially lands on tails, we flip again. Determine the probability that exactly 3 of the 10 coins eventually land on heads. The probability of initially getting exactly i heads out of 10 is $\binom{10}{i}/2^{10}$. The probability of getting $3-i$ heads out of $10-i$ is $\binom{10-i}{3-i}/2^{10-i}$. Therefore, the probability that exactly 3 of the 10 coins eventually land on heads is

$$\begin{aligned}
\sum_{i=0}^{3} \frac{\binom{10}{i}}{2^{10}} \cdot \frac{\binom{10-i}{3-i}}{2^{10-i}} &= \frac{\binom{10}{0}}{2^{10}} \cdot \frac{\binom{10}{3}}{2^{10}} + \frac{\binom{10}{1}}{2^{10}} \cdot \frac{\binom{9}{2}}{2^{9}} + \frac{\binom{10}{2}}{2^{10}} \cdot \frac{\binom{8}{1}}{2^{8}} + \frac{\binom{10}{3}}{2^{10}} \cdot \frac{\binom{7}{0}}{2^{7}} \\
&= \frac{\binom{10}{3}}{2^{20}} + \frac{\binom{10}{1}\binom{9}{2}}{2^{19}} + \frac{\binom{10}{2}\binom{8}{1}}{2^{18}} + \frac{\binom{10}{3}}{2^{17}} \\
&\approx 0.0309
\end{aligned}$$

4 The Project

Part I:
Consider the experiment of rolling five dice all at once. Clearly define the 6 different results (Three of a kind, Four of a kind, Full House, Small Straight, Large Straight, and YAHTZEE) as in the official game of Yahtzee.

For each result, explain how to calculate the number of ways to yield that result. Note that the different outcomes are not mutually exclusive. For example, a Full House can also be scored as a Three of a Kind. Make sure to account for this in your final calculations. Summarize your values in a table, including the probability of each result. To help with your calculations, you may assume that the five dice are each of a different size (or are each colored differently, etc.) so that they are distinguishable.

Part II:
If you've ever played the game of Yahtzee, you'll know that often times, the last line to be filled in is YAHTZEE. For this part of the project, you will calculate the probability of rolling Yahtzee on a single turn. To this end, suppose that after you roll five dice, you are allowed to select any of the five dice and roll them again. At which point, you may select any of the five dice and roll them for a third time. The following calculations will allow us to calculate the probability of rolling YAHTZEE after any of the three rolls.

First, consider the situation where you have exactly i dice of the same value and you re-roll the other $5-i$ dice. Assuming that you have i dice of the same value that you are not going to re–roll, let $p_{i,j}$ denote the probability that you end up with exactly j dice of the same value *after* you re–roll the other $5-i$ dice. Note that if $j < i$ then $p_{i,j} = 0$ since you certainly have at least i dice all of the same value after you roll the other $5-i$ dice.

For example, suppose that you first roll five dice resulting in the values $(2,3,3,4,6)$. You decide to keep the two 3s and re–roll the other three dice. The quantity $p_{2,3}$ represents the probability of having exactly three dice of the same value after this second roll. Note that this could happen in one of two ways. You could either roll exactly one additional 3 on the second roll or have all three dice showing the same value other than 3.

Calculate all of the entries in the following matrix T, which is called a *transition matrix.*

$$T = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} & p_{1,5} \\ 0 & p_{2,2} & p_{2,3} & p_{2,4} & p_{2,5} \\ 0 & 0 & p_{3,3} & p_{3,4} & p_{3,5} \\ 0 & 0 & 0 & p_{4,4} & p_{4,5} \\ 0 & 0 & 0 & 0 & p_{5,5} \end{bmatrix}$$

Calculate all of the values in the above matrix and make sure to give a detailed explanation of how to calculate the values $p_{2,2}$, $p_{2,3}$, $p_{2,4}$, and $p_{2,5}$. You do not need to explain how to calculate the other entries. Once you have calculated all of the entries in the above matrix, let me know and I will help you calculate T^3. The entry in the first row and fifth column of T^3 is the probability of getting YAHTZEE using at most three rolls.

Part III:
For the final portion of the project, we will focus on how to maximize your score on CHANCE. Suppose that on your last turn, the only item you have left to fill is CHANCE. What strategy should you use to maximize your score? Explain how to decide which dice you should keep after each roll and which dice you should re–roll.

In order to determine the best strategy, we need to understand how to calculate the expected value of a random event. Suppose that the sample space of a random event is $\{x_1, x_2, \ldots, x_k\}$ and that the probability of each outcome is $\{p_1, p_2, \ldots, p_k\}$, respectively. That is, the outcome of our experiment is x_i with probability p_i, for each $i = 1, 2, \ldots, k$. Then the *expected value* of our random event is defined to be the following quantity:

$$\sum_{i=1}^{k} x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3 + \cdots + x_k p_k.$$

This quantity represents a theoretical average value. In other words, if we were to perform our experiment many times, then the average value of our actual outcomes should be close to the expected value.

To begin, consider rolling a single die at most three times. Determine a strategy that achieves the maximum expected value of $4.\overline{6}$ for the final value of this single die. The expected value of the total of all five dice would then be five times the expected value of a single die. In other words, the maximum expected score for CHANCE is $23.\overline{3}$.

Explain why this is the maximum expected value. To do so, it may be helpful to consider the expected value of the die when you are only allowed to roll the die once and when you are allowed to roll the die at most two times.