Math 310: Elementary Combinatorics

Sections 1 & 2

Penn State University

Spring 2009

Group Project: Perfect Coverings

1 Introduction

The first combinatorial objects that we discussed this semester were perfect coverings of a chessboard with dominoes. In this project, we will again turn our attention to perfect coverings by studying the following variations on this theme:

- Perfect coverings of a 3–by–n board with dominoes.
- Perfect coverings of an Aztec diamond with dominoes.
- Perfect coverings of a 1–by–n board with squares of various colors.

2 Rules and Regulations

Students will work in groups consisting of no more than 4 people. Each group will submit one typed report, including a cover page with project title and names of group members. Your report should have an introduction, giving a brief description of the project and any other information or basic definitions/assumptions you will be using that are relevant to the entire project. Each of the three parts of the project should also include a brief introduction describing the specific experiment and any other information that is relevant to that part of the project.

The group report will be graded out of 75 points (15% of your overall grade). Each project consists of three parts (see below). Part I is worth 20 points, Part II is worth 20 points and Part III is worth 20 points. Each of these parts will be graded for mathematical detail and accuracy. The remaining 15 points will be awarded based on presentation (clarity and consistency of writing style, grammatical correctness, appropriate use of notation, terminology, and examples, etc.)

Make sure to tell me who is in your group and which project you will be working on by **Friday, March 27**. During the week of **April 13 through April 17** or before, your group must show me a rough draft of your report, at which point I will make any necessary comments and/or suggestions. **The deadline to submit your completed project is Monday, April 27.**

3 The Project

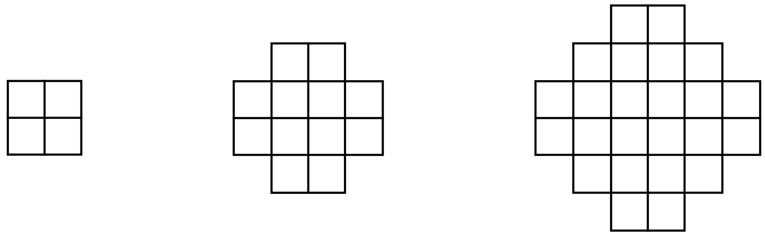
Part I:

Find and prove a recurrence relation for g_n, the number of perfect coverings of a 3–by–$2n$ chessboard with dominoes. To do so, group perfect coverings together based on the position of the first horizontal domino (from top to bottom) that crosses the vertical line between the second and third columns, if any such domino exists. Make sure to explain the value of g_n for any value of n to which your recurrence relation does not apply. Use your recurrence relation to find a generating function as well as an explicit formula for g_n.

To complete our study of perfect coverings of a 3–by–$2n$ board, find a formula for the number of perfect coverings based on the number of vertical dominoes used. More specifically, show that every perfect covering of a 3–by–$2n$ chessboard must use an even number of vertical dominoes. Use this fact to find a formula for the number of perfect coverings of a 3–by–$2n$ chessboard that use $2k$ vertical dominoes for $0 \leq k \leq n$.

Part II:

An Aztec diamond of size n is a chessboard consisting of $2n$ rows of lengths $2, 4, 6, \ldots, 2n, 2n$, $2n-2, 2n-4, \ldots, 4, 2$, from top to bottom. Each row of squares is centered on a vertical line that runs through the middle of the board. The following diagrams illustrate Aztec diamonds of size $n = 1, 2$, and 3.



The goal of this portion of the project is to conjecture a formula for a_n, the number of perfect coverings of an Aztec diamond of size n. Compute a_n for $n = 0, 1, 2, 3$. Give a detailed explanation as to how you have determined these values, without necessarily listing all of the perfect coverings. Find the pattern for the number of perfect coverings of an Aztec diamond of size n. Hint: Each value of a_n can be written as a power of 2.

Part III:

For this final portion of the project, we will study perfect coverings of 1–by–n chessboards using squares of various colors.

We begin with a brief introduction to exponential generating functions. Suppose that we have disjoint multisets A and B and let a_n be the number of ways to select an n–permutation from A and let b_n be the number of ways to select an n–permutation from B. Then the exponential generating functions associated with a_n and b_n are given by:

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \qquad \text{and} \qquad \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$$

The advantage to using exponential generating functions in this situation is based on the combinatorial interpretation of multiplying two exponential generating functions. In particular, consider the following calculations.

$$\begin{aligned}
\left(\sum_{r=0}^{\infty} a_r \frac{x^r}{r!}\right)\left(\sum_{s=0}^{\infty} b_s \frac{x^s}{s!}\right) &= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \frac{1}{r!(n-r)!} a_r b_{n-r}\right) x^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \frac{n!}{r!(n-r)!} a_r b_{n-r}\right) \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \binom{n}{r} a_r b_{n-r}\right) \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}
\end{aligned}$$

where

$$c_n = \sum_{r=0}^{n} \binom{n}{r} a_r b_{n-r}$$

Note that c_n is the number of n-permutations from the union of A and B. For more information and examples regarding exponential generating functions, see Chapter 7 of our textbook.

And now back to perfect coverings. Specifically, we will be investigating perfect coverings of a 1–by–n board using red, blue, green, and yellow squares. For the purposes of using exponential generating functions, we can imagine that we have four multisets, each one consisting of an infinite number of identical squares. Each multiset will correspond to one of the four colors: red, blue, green, and yellow.

For each of the following types of perfect coverings, find the corresponding exponential generating function and use it get an explicit formula for the number of perfect coverings of a 1–by–n board of the given type.

1. Perfect coverings with at least one red square, at least two blue squares, and any number of green and yellow squares.

2. Perfect coverings with at most two red squares, an even number of yellow squares, at least one green square, and any number of blue squares.

3. Perfect coverings with an even number of blue squares, an odd number of green squares, at least one yellow square, and at most one blue square.

The following identities involving exponential functions may prove useful:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4}! + \cdots$$

$$e^{ax} = \sum_{n=0}^{\infty} a^n \frac{x^n}{n!} = 1 + ax + a^2 \frac{x^2}{2!} + a^3 \frac{x^3}{3!} + a^4 \frac{x^4}{4!} + \cdots$$

$$\frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

$$\frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$