Permutations

1. How many ways are there to seat five people in a row of five chairs?

2. How many ways are there to seat five people in a row of eight chairs?

3. How many ways are there to seat eight people in a row of five chairs? In this case, three people will have to remain standing in no particular arrangement.

Consider a set S with exactly n distinct elements. A *permutation* of S is a linear rearrangement (i.e., an ordered list) of all n elements of S. For example, if $S = \{1, 2, 3, 4, 5\}$, then

$$(3, 5, 2, 4, 1) \quad \text{and} \quad (2, 4, 1, 3, 5)$$

are both permutations of S. An r*-permutation* of S is a linear rearrangement of exactly r elements of S. For example,

$$(7, 2, 4, 8, 1) \quad \text{and} \quad (5, 2, 8, 3, 6)$$

are both 5-permutations of $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

4. What do the first three questions have to do with permutations and/or r-permutations? In particular, how do the two specific permutations shown above correspond to seating five people in a row of five chairs? How do the two 5-permutations shown above correspond to seating five people in a row of eight chairs? How do the two 5-permutations shown above correspond to seating eight people in a row of five chairs?

5. Use the product rule to determine the number of permutations of an n-element set? How many ways are there to seat n people in a row of n chairs?

6. How many ways are there to seat 15 people in 3 rows of 5 chairs each? Think of the chairs as being in a rectangular arrangement so that there is a first, second and third row of chairs. How is this different from seating 15 people in a single row of 15 chairs?

7. Use the product rule to determine a formula for $P(n, r)$, the number of r-permutations of an n-element set? How many ways are there to seat r people in a row of n chairs for $0 \leq r \leq n$? How many ways are there to seat n people in a row of r chairs for $0 \leq r \leq n$?

8. Write your formula for $P(n, r)$ as a ratio of factorials. Recall that $n!$ (read "n factorial") is the product of the first n positive integers, that is $n! = 1 \times 2 \times \cdots \times n$. Explain your formula using the quotient rule.

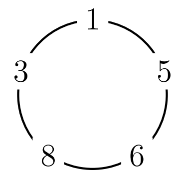
9. What does $P(n, n)$ represent and how does your formula from the previous problem suggest $0!$ should be defined? How many ways are there to seat zero people in a row of zero chairs?

10. Suppose you have 5 boys and 5 girls. How many ways are there to seat all 10 children in a row so that no boy sits next to another boy and no girl sits next to another girl?

11. We saw in class that there are $\frac{(2n)!}{2^n n!}$ ways to pair off $2n$ people into n groups of 2. Explain this formula using permutations and the quotient rule.

12. How many ways are there to seat five people at a circular table with five chairs? Keep in mind that there is no way to distinguish between the chairs at a *circular* table until there is at least one person sitting at the table. How many ways are there to seat eight people at a circular table with five chairs?

A *circular r-permutation* of an n-element set is a circular rearrangement of exactly r elements. For example,



is a circular 5-permutation of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

13. We can form a circular r-permutation from an r-permutation by writing the entries of the r-permutation on a circle in a clockwise fashion. For example, the 5-permutations $(5, 6, 8, 3, 1)$ and $(8, 3, 1, 5, 6)$ both correspond to the circular 5-permutation shown above. How many different 5-permutations correspond to the circular 5-permutation shown above?

14. How many different r-permutations correspond to the same circular r-permutation?

15. Use your formula for $P(n, r)$ and the quotient rule to find a formula for the number of circular r-permutations. How many ways are there to seat n people at a circular table with r chairs, for $0 \leq r \leq n$?

16. Recall Problem 10. What if the children are sitting at a circular table?

17. How many different 5-beaded necklaces can you make if you have 8 beads, each of a different color? How is this different from seating 8 people at a circular table with 5 chairs? What can be done with our necklace that cannot/should not be done with a table?

Challenge Problems

18. Recall Problem 16. Suppose that these children are made up of 5 brother/sister pairs. How many ways are there to seat the children at a circular table as before, but with the additional requirement that no two siblings sit next to each other?

19. Suppose you have 2 red beads, 2 blue beads, 2 green beads, and 2 yellow beads. How many different 4-beaded necklaces can you make? 5-beaded necklaces?