Integer Partitions and Generating Functions

The following problems will guide you through the process of finding a few of the standard generating functions for p_n, the number of partitions of n.

1. Write down a generating function for the number of partitions of n with largest part *at most* 6. In other words, find a generating function for the number of nonnegative integer solutions to
$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 = n$$
where x_i is the number of parts equal to i in the partition. Write down the generating function for the number of partitions of n with largest part *at most* k.

2. Find a generating function for p_n, the number of partitions of n, in the form of an infinite product. Hint: Let k go to infinity in your answer to the previous problem. Using your list of partitions from the previous worksheet, write down the first eight terms of the series expansion of the generating function for p_n.

3. Write down a generating function for partitions of n with largest part *equal to* 6 (i.e., $x_6 \geq 1$). Write down the generating function for the number of partitions of n with largest part *equal to* k.

4. Write down a generating function for the number of partitions of n with *at most* k parts. Write down the generating function for $p_{n,k}$, the number of partitions of n with *exactly* k parts.

5. Use your generating function for $p_{n,k}$ to write down a generating function for p_n in the form of an infinite sum (i.e., sum over all possible values of k). Using your answer to Problem 2, what identity relating two generating functions have you discovered?

6. Write down a generating function for the number of partitions of n that contain a $k \times k$ Durfee square. Hint: Decompose such a partition λ into its Durfee square and the partitions μ and γ, as described in Problem 9 of the previous worksheet.

7. By summing your answer to the previous problem over all possible values of k, you now have yet another way to express the generating function for p_n as an infinite sum. Again using your answer to Problem 2, what identity relating two generating functions have you discovered?

The following problems will guide you through the process of finding a few of the standard generating functions for the number of partitions *with distinct parts*.

8. Write down a generating function for the number of partitions of n with distinct parts and largest part is at most 7. Write down a generating function for the number of partitions of n with distinct parts and largest part is at most k.

9. Write down a generating function for partitions of n with distinct parts in the form of an infinite product. Hint: Let k go to infinity in your answer to the previous problem. Using information from the previous worksheet, write down the first eight terms of the series expansion of this generating function.

10. Write down a generating function for partitions of n with exactly k distinct parts. Hint: Start with the generating function for partitions with at most k parts (i.e., nonnegative integer solutions to $x_1 + 2x_2 + \cdots + kx_k = n$) and add the restriction that $x_i \geq 1$ for all $1 \leq i \leq k$. Explain why this results in a partition with distinct parts. Think about what x_i represents in this context. It is not the number of parts equal to i as described in Problem 1.

11. Use your answer to the previous problem to write down another generating function for partitions with distinct parts in the form of an infinite sum. What identity relating two generating functions have you discovered?

Using similar techniques to those that you used to answer the previous problems, try to find generating functions for each of the following types of partitions.

12. Find a generating function in the form of an **infinite product** for each of the following types of partitions.
    (a) partitions of n with only odd parts.
    (b) partitions of n with only distinct odd parts (no even parts).
    (c) partitions of n with no part a multiple of 3.
    (d) partitions of n where no part is a perfect square.
    (e) partitions of n with no part appearing 3 or more times.
    (f) partitions of n where the part i appears at most $i-1$ times (i.e., 1 does not appear, 2 appears at most once, 3 appears at most twice, etc.).
    (g) partitions of n with parts that are congruent to ± 1 modulo 5 (i.e., $1, 4, 6, 9, 11, 14, 16, ...$).
    (h) partitions of n with parts that are congruent to ± 2 modulo 5 (i.e., $2, 3, 7, 8, 12, 13, 17, 18, ...$).

13. Find a generating function in the form of an **infinite sum** for each of the following types of partitions.
    (a) partitions of n with parts that differ by at least two.
    (b) partitions of n with parts that differ by at least two and the smallest part is at least two.

14. Find a generating function for the number of partitions of n that are self-conjugate and have a k-by-k Durfee square. Hint: How many dots are contained in the k-by-k Durfee square? How can you account for the fact that each column of dots to the right of the Durfee square corresponds to a row of dots above the Durfee square?

15. Find a generating function for the number of partitions of n that are self-conjugate. Your answer should be in the form of an infinite sum.

16. In the previous worksheet, you discovered a number of theorems about partitions. State each of these theorems as an identity relating two generating functions. Try to prove as many of these identities as you can.