Math 310: Elementary Combinatorics

Penn State University

Spring 2009

Sections 1 & 2

Integer Partitions

1. Create a list of partitions of n for $0 \leq n \leq 7$. Compute p_n, the number of partitions of n.

n	partitions of n	p_n
0		
1		
2		
3		
4		
5		
6		
7		

2. Use the information you collected above to fill in the following tables.

In row n and column k, place the number of partitions of n into *exactly k parts*.

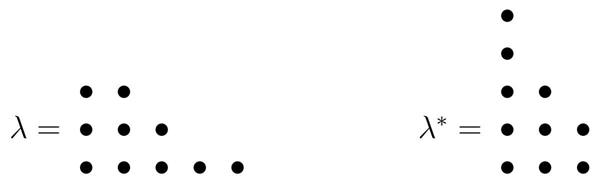
	1	2	3	4	5	6	7
1							
2							
3							
4							
5							
6							
7							

In row n and column k, place the number of partitions of n with *largest part equal to k*.

	1	2	3	4	5	6	7
1							
2							
3							
4							
5							
6							
7							

3. What do you notice about the number of partitions of n into exactly k parts versus the number of partitions of n with largest part equal to k? Can you explain this behavior?

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition. We define the *conjugate* partition of λ, denoted by $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_l^*)$ where λ_i^* is the number of dots in the ith column (from left to right) of the Ferrers diagram of λ. For example, if $\lambda = (5, 3, 2)$ then $\lambda^* = (3, 3, 2, 1, 1)$, as illustrated below.



Note that the Ferrers diagram of λ^* is simply obtained by reflecting the Ferrers diagram of λ about the line $y = x$.

4. Find the conjugate of each of the following partitions.

   (a) $\lambda = (5)$ $\qquad \lambda^* =$ ____________________

   (b) $\lambda = (7, 4)$ $\qquad \lambda^* =$ ____________________

   (c) $\lambda = (6, 5, 3, 3, 2)$ $\qquad \lambda^* =$ ____________________

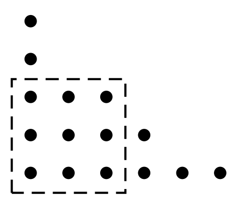
   (d) $\lambda = (5, 5, 4, 2, 2, 1)$ $\qquad \lambda^* =$ ____________________

5. If the largest part of λ is 7, what can you say about the number of parts in λ^*? If the number of parts in λ is 11, what can you say about the largest part in λ^*?

6. If you weren't able to justify your answer to Problem 3, can you now?

7. What can you say about the number of partitions of n with largest part *at most* k versus the number of partitions of n with *at most* k parts? Can you justify your answer?

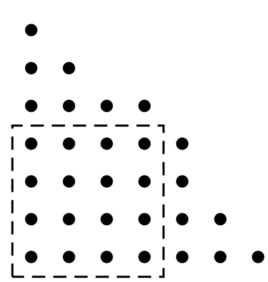
The Durfee square of the partition λ is the largest square that can be imbedded in the Ferrers diagram of λ. More formally, the size of the Durfee square is the largest value i such that $\lambda_i \geq i$. For example, the partition $\lambda = (6, 4, 3, 1, 1)$ has a 3×3 Durfee square associated with it, as illustrated below.



8. Find the size of the Durfee square associated with each of the partitions λ and λ^* given in Problem 4.

9. For a given partition λ, let μ be the partition whose Ferrers diagram lies to the right of the Durfee square in λ and let γ be the partition whose Ferrers diagram lies above the Durfee square in λ. In the above example, $\mu = (3, 1)$ and $\gamma = (1, 1)$. In general, what can you say about the number of parts of μ and the largest part of γ when λ contains a $k \times k$ Durfee square?

A partition λ is said to be *self-conjugate* if $\lambda^* = \lambda$. In other words, λ and its conjugate partition λ^* are the same partition. For example, the partition $(7, 6, 5, 5, 4, 2, 1)$ is self-conjugate. The Ferrers diagram of a self-conjugate partition can be decomposed in the following manner based on its Durfee square:



Notice that the partition to the right of the Durfee square, namely (3,2,1,1), and the partition above the Durfee square, namely (4,2,1), are conjugates of each other.

10. Determine the number of partitions of 20 that are self-conjugate.

11. Determine the number of partitions of 21 that are self-conjugate.

Many of the most famous results in the theory of partitions can be stated in the following manner:

"The number of partitions of n that __________ is equal to the number of partitions of n that __________."

To help discover several results of this form, compute the first eight terms of each of the following sequences of partitions and see if you can find any pairs of sequences that are equal. If so, state your result in the form described above (i.e., "the number of partitions of n that __________ is equal to...").

12. Use the data you collected in Problem 1 to write down the first 8 terms (i.e., $0 \leq n \leq 7$) of each of the following sequences of partitions.

    (a) partitions of n with distinct parts.

    (b) partitions of n with only odd parts.

    (c) partitions of n with only distinct odd parts (no even parts).

    (d) partitions of n with no part a multiple of 3.

    (e) partitions of n where no part is a perfect square.

    (f) partitions of n with no part appearing 3 or more times.

    (g) partitions of n where the part i appears at most $i-1$ times (i.e., 1 does not appear, 2 appears at most once, 3 appears at most twice, etc.).

    (h) partitions of n with parts that are congruent to ± 1 modulo 5 (i.e., $1, 4, 6, 9, 11, 14, 16, ...$).

    (i) partitions of n with parts that are congruent to ± 2 modulo 5 (i.e., $2, 3, 7, 8, 12, 13, 17, 18, ...$).

    (j) partitions of n with parts that differ by at least two.

    (k) partitions of n with parts that differ by at least two and the smallest part is at least two.

    (l) partitions of n that are self-conjugate.