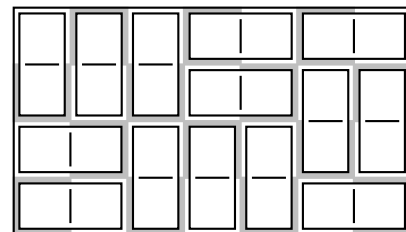


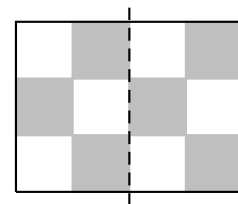
Perfect Coverings of Chessboards

A *perfect covering* of a chessboard with dominoes (i.e., a rectangle that covers two adjacent squares of a chessboard) is an arrangement of identical dominoes such that every square of the chessboard is covered by exactly one domino.



The following questions guide you through the process of counting the number of perfect coverings for chessboards of varying dimensions.

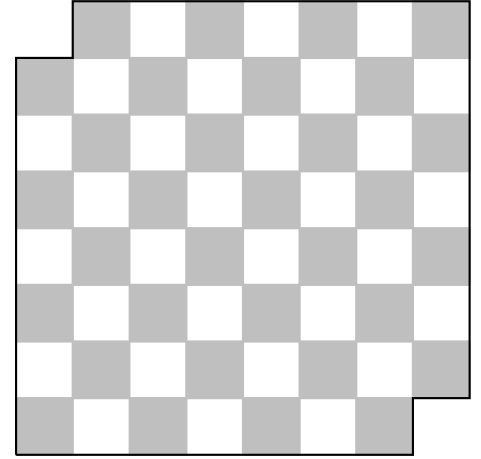
1. Find a formula for G_n , the number of perfect coverings of a 1-by- n chessboard with dominoes, for all positive integers n .
2. Compute F_n , the number of perfect coverings of a 2-by- n chessboard with dominoes, for $n = 1, 2, 3, 4, 5$.
3. Explain how to use perfect coverings of a 2-by-4 chessboard and perfect coverings of a 2-by-5 chessboard to construct all of the perfect coverings of a 2-by-6 chessboard without listing them all out. Compute F_6 .
4. Based on the data you've collected so far, find a relationship between F_n , F_{n-1} and F_{n-2} for all $n \geq 3$. Use this relationship to compute F_{10} . Can you prove this relationship you discovered?
5. How should we define F_0 , the number of perfect coverings of a 2-by-0 chessboard, so that the relationship you found in the previous question holds for F_0 , F_1 and F_2 ? Does this definition seem reasonable? Explain.
6. Compute the number of perfect coverings of a 3-by-3 chessboard.
7. Find a necessary and sufficient condition on the integers m and n so that there exists at least one perfect covering of an m -by- n chessboard. Can you prove it? Recall that "necessary", in this case, means that if m and n do *not* have this condition, then there is no perfect covering. "Sufficient" means that if m and n do have this condition, then there is at least one perfect covering.
8. Compute the number of perfect coverings of a 3-by-4 chessboard. It may help to group perfect coverings together according to a specific property. For example, group perfect coverings together based on the position of the first horizontal domino (from top to bottom) that crosses a vertical line through the middle of the board, if any such domino exists.
9. Compute the number of perfect coverings of a 4-by-4 chessboard.
10. For the previous two questions, how many of these perfect coverings are still different if you are allowed to rotate the chessboard?



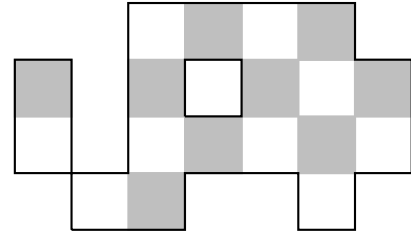
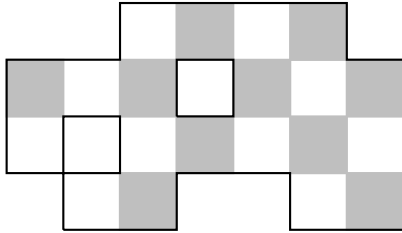
For 3-by-4 chessboards, you can only rotate the board 180 degrees to get back to a 3-by-4 board. Try to find all of the perfect coverings that are unchanged after rotating the board 180 degrees. The remaining coverings can be broken up into pairs that are equivalent to each other.

For 4-by-4 chessboards, you can rotate the board 90 degrees at a time. Try to find all of the perfect coverings that are unchanged after rotating the board 90 or 180 degrees. The remaining coverings can be broken up into groups of four perfect coverings that are equivalent to each other.

11. Is it possible to have a perfect covering of an 8-by-8 chessboard where opposite corners have been removed from the board? Justify your answer.



12. A pruned chessboard is a chessboard obtained by removing any number of squares from a rectangular chessboard without separating the board into two or more smaller chessboards. For example, the board on the left is a pruned chessboard while the board on the right is not pruned since the two left-most squares do not have an edge in common with the rest of the board.



Is it possible to have a pruned chessboard that has an even number of squares but does not have a perfect covering? Is it possible to have a pruned chessboard that has the same number of white and black squares but does not have a perfect covering? Justify your answers.