Combinations

A *binary sequence of length* n is an n digit number where each digit is either a zero or a one. For example, the eight binary sequences of length 3 are listed below:

$$000 \quad 001 \quad 010 \quad 011 \quad 100 \quad 101 \quad 110 \quad 111$$

1. Describe how to use binary sequences of length 3 to construct all the binary sequences of length 4. How many binary sequences are there of length 4? Describe how to use binary sequences of length $n-1$ to construct all the binary sequences of length n. How many binary sequences are there of length n?

2. How many binary sequences are there of length 5 that have exactly r ones, for $r = 0, 1, 2, 3, 4, 5$? Try to answer this question without writing down all the binary sequences of length 5.

Consider a set S with exactly n distinct elements. A *combination* of S is simply an unordered collection of elements of S (i.e., a subset of S). For example, all eight of the combinations of $S = \{1, 2, 3\}$ are listed below:

$$\{\} \quad \{1\} \quad \{2\} \quad \{1,2\} \quad \{3\} \quad \{1,3\} \quad \{2,3\} \quad \{1,2,3\}$$

An r-*combination* of S is an unordered collection of r elements of S (i.e., an r-element subset of S). For example, the 2-combinations of $\{1, 2, 3\}$ are

$$\{1,2\} \quad \{1,3\} \quad \{2,3\}$$

3. Describe how to use combinations of $\{1, 2, 3\}$ to construct all the combinations of $\{1, 2, 3, 4\}$. How many combinations are there of $\{1, 2, 3, 4\}$? Describe how to use combinations of $\{1, 2, \ldots, n-1\}$ to construct all the combinations of $\{1, 2, \ldots, n-1, n\}$. How many combinations are there of $\{1, 2, \ldots, n-1, n\}$?

4. How many r-combinations are there of $\{1, 2, 3, 4, 5\}$, for $r = 0, 1, 2, 3, 4, 5$? Try to answer this question without writing down all the combinations of $\{1, 2, 3, 4, 5\}$.

5. At this point, you should notice a striking similarity between Questions 1 & 2 and Questions 3 & 4. What is the connection between binary sequences and combinations? In particular, explain how to convert a binary sequence of length n into a combination of $\{1, 2, \ldots, n\}$ and vice versa. If you have a binary sequence of length n with r ones, what can you say about the corresponding combination?

6. Note that the only difference between an r-permutation and an r-combination is whether or not the elements are ordered. In particular, the 4-permutations $(5, 2, 6, 3)$ and $(2, 5, 6, 3)$ correspond to the 4-combination $\{2, 3, 5, 6\}$ in the sense that they are both reorderings of the elements of $\{2, 3, 5, 6\}$. How many different 4-permutations correspond to the 4-combination $\{2, 3, 5, 6\}$? How many different r-permutations correspond to *any* given r-combination?

7. The number of r-combinations of an n-element set is denoted by $\binom{n}{r}$, read "n choose r", since it can be interpreted as the number of ways to choose r objects from a collection of n distinct objects. Use your formula for $P(n, r)$ and your answer to the previous question to find a formula for $\binom{n}{r}$.

8. How many binary sequences of length n have exactly r ones where $0 \leq r \leq n$?

9. Use the sum rule and your answer to the previous question to count the total number of binary sequences of length n. In other words, partition the binary sequences of length n according to the number of ones in each sequence.

10. Compare your answers from Problems 1 and 9. What theorem have you just discovered (and proved)?

11. Compute $\binom{7}{1}$ and $\binom{7}{6}$. Compute $\binom{7}{2}$ and $\binom{7}{5}$. Compute $\binom{7}{3}$ and $\binom{7}{4}$. What do you notice? What can you say about $\binom{n}{r}$ and $\binom{n}{n-r}$? Give a combinatorial explanation for this behavior.

12. How many 2-combinations of $\{1, 2, 3, \ldots, 13\}$ have 8 as its largest element? How many 2-combinations of $[n+1] = \{1, 2, 3, \ldots, n, n+1\}$ have r as its largest element, for each $r = 1, 2, \ldots, n+1$?

13. Simplify $\binom{n+1}{2}$ as much as possible using the formula you found in Problem 7. Recall that $\binom{n+1}{2}$ represents the number of 2-combinations of $[n+1]$ (or the number of ways to select 2 objects from a collection of $n+1$ distinct objects or the number of binary sequences of length $n+1$ with exactly 2 ones). Use the sum rule and your answer to the previous question to count the number of 2-combinations of $[n+1]$. In other words, partition the 2-combinations of $[n+1]$ according to the largest number in the combination. What theorem have you just re-discovered?

14. We know that there are $\frac{(2n)!}{2^n n!}$ ways to pair off $2n$ people into n groups of 2. By using your formula for $\binom{n}{r}$, verify that $\frac{(2n)!}{2^n n!}$ is equal to

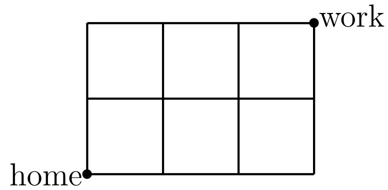
$$\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}/n!.$$

Give a combinatorial explanation for this formula using combinations and the quotient rule.

15. How many ways are there to group mn people into n groups of m? Try to answer this question in as many different ways as possible. Think about all the different formulas we have encountered that yield the number of ways to pair off $2n$ people into n groups of 2.

16. How many integers between 1 and 1,000,000 have exactly 3 odd digits? What if all 3 odd digits must be different?

17. Suppose that you live 3 blocks west and 2 blocks south of where you work.



How many different routes are there for you to take to work from home, assuming that you always travel to the north or east one block at a time?

18. Suppose that you live m blocks west and n blocks south of where you work. How many different routes are there for you to take to work from home, assuming that you always travel to the north or east one block at a time?

Hint: Use binary sequences to write down possible routes. Use a zero to mean that you travel one block east and a one to mean that you travel one block north. How many zeroes are there in your binary sequences? How many ones? How many different binary sequences have this same property? Does each sequence correspond to a different route?