Math 310: Elementary Combinatorics

Sections 1 & 2

Penn State University

Spring 2006

Group Project: Combinatorics and Games of Chance

1 Introduction

There is a very strong connection between the methods of counting that we have studied thus far and the foundations of probability theory. Games of chance, be it flipping a coin, picking numbers out of a hat, or your favorite card/dice game, can all be analyzed from a purely combinatorial point of view. The goal of this project is to do just that.

2 Rules and Regulations

Students will work in groups consisting of no more than 4 people. Each group will complete an analysis of one of the three games of chance described below. Which one is entirely up to you and the other people in your group. Each group will submit one typed report, including a cover page with project title and names of group members.

Your report should have an introduction, giving a brief description of the game being analyzed and any other relevant information or basic definitions/assumptions you would like to make. Also make sure to include any information specific to each project as described below.

The group report will be graded out of 50 points (10% of your overall grade). Each project consists of three parts (see below). Part I is worth 15 points, Part II is worth 10 points and Part III is worth 10 points. Each of these parts will be graded for mathematical detail and accuracy. The remaining 15 points will be awarded based on presentation (clarity of writing style, grammatical correctness, appropriate use of notation and examples, etc.)

Make sure to tell me who is in your group and which project you will be working on by Friday, March 24. By Friday, April 14, your group must show me a rough draft of your report, at which point I will make any necessary comments and/or suggestions. **The deadline to submit your completed project is Wednesday, April 26.**

3 Background on Probability

Suppose we conduct an experiment that has n distinct outcomes where each outcome is equally likely. That is, each outcome occurs with probability $1/n$. For example, selecting 5 cards from a deck of 52, rolling 5 dice or selecting 5 numbers between 1 and 55 would all qualify as experiments where each outcome (for each individual experiment) is equally likely. Let S be the collection of all possible outcomes from our experiment. In probability theory, S is referred to as the *sample space*.

Any combination (i.e. subset) of S is referred to as an *event*. For example, selecting 5 cards all of the same suit would be an event. If an event A consists of r distinct outcomes from our experiment, then the probability of event A, denoted by $P(A)$, is given by

$$P(A) = \frac{\text{number of outcomes in the event } A}{\text{number of outcomes in the sample space } S} = \frac{r}{n}.$$

Consider the following experiment. Flip a coin 10 times and record an "H" if it lands on heads or a "T" if it lands on tails for each coin flip. Our sample space S consists of all sequences of H's and T's of length 10 (much like binary sequences except that we are using H's and T's instead of 0's and 1's). Let A be the event that we record exactly 8 H's in our experiment. Therefore

$$P(A) = \frac{\binom{10}{8}}{2^{10}} \approx 0.04395$$

In other words, this event occurs approximately 4.395% of the time.

The function $P(A)$ defined above satisfies properties very similar to the Sum and Product Rules that we have been using for counting. In particular, if the outcomes in event A can be partitioned into events $A_1, A_2, \ldots, A_k$ (i.e. $\{A_1, A_2, \ldots, A_k\}$ is a set partition of A) then

$$P(A) = P(A_1) + P(A_2) + \cdots + P(A_k).$$

On the other hand, if each outcome in A can be constructed using the product rule, that is to say that each event can be uniquely constructed from a series of experiments (A_i is an event of the ith experiment) and the number of outcomes in A_i is independent of previous outcomes, then

$$P(A) = P(A_1) \times P(A_2) \times \cdots \times P(A_k).$$

For example, the probability of flipping a coin 10 times and recording 7 or 8 heads would be given by

$$P(7 \text{ H's out of 10 coin flips}) + P(8 \text{ H's out of 10 coin flips}) = \frac{\binom{10}{7}}{2^{10}} + \frac{\binom{10}{8}}{2^{10}}.$$

The probability of flipping a coin 10 times and recording 4 heads and then flipping a coin 15 times and recording 9 heads would be given by

$$P(4 \text{ H's out of 10 coin flips}) \times P(9 \text{ H's out of 15 coin flips}) = \frac{\binom{10}{4}}{2^{10}} \times \frac{\binom{15}{9}}{2^{15}}.$$

One last example. Suppose that we flip 10 distinct coins and any coin that initially lands on tails, we flip again. Determine the probability that exactly 3 of the 10 coins eventually land on heads. The probability of initially getting exactly i heads out of 10 is $\binom{10}{i}/2^{10}$. The probability of getting $3-i$ heads out of $10-i$ is $\binom{10-i}{3-i}/2^{10-i}$. Therefore, the probability that exactly 3 of the 10 coins eventually land on heads is

$$\begin{aligned}\sum_{i=0}^{3} \frac{\binom{10}{i}}{2^{10}} \cdot \frac{\binom{10-i}{3-i}}{2^{10-i}} &= \frac{\binom{10}{0}}{2^{10}} \cdot \frac{\binom{10}{3}}{2^{10}} + \frac{\binom{10}{1}}{2^{10}} \cdot \frac{\binom{9}{2}}{2^{9}} + \frac{\binom{10}{2}}{2^{10}} \cdot \frac{\binom{8}{1}}{2^{8}} + \frac{\binom{10}{3}}{2^{10}} \cdot \frac{\binom{7}{0}}{2^{7}} \\ &= \frac{\binom{10}{3}}{2^{20}} + \frac{\binom{10}{1}\binom{9}{2}}{2^{19}} + \frac{\binom{10}{2}\binom{8}{1}}{2^{18}} + \frac{\binom{10}{3}}{2^{17}} \\ &\approx 0.0309\end{aligned}$$

4 Games of Chance

4.1 Poker

Part I:

Consider the experiment where you are dealt a hand of 5 cards from a standard deck of 52 cards. Clearly define the 9 different possible types of hands of Poker (High Card, One Pair, Two Pair, Three of a Kind, Four of a Kind, Full House, Flush, Straight and Straight Flush). For each type of hand, explain how to calculate the number of hands of poker of that type. Summarize your results in a table, including the probability of being dealt each type of hand, ordering the hands from least likely to most likely.

Part II:
Now suppose that after you are dealt five cards, you may discard any of these cards and be dealt new cards to replace them from the original deck. Calculate the probability of getting a Full House either initially or after you discard some of your cards. Make sure to explain your general strategy for which cards you will keep and which cards you will discard. Your strategy should maximize the probability of getting a Full House. Alternatively, you could calculate the probability of getting a Four of a Kind or a Flush.

Part III:
Consider the following variation of Poker called Texas Hold 'Em. First, each player is dealt two cards from a standard deck of 52 cards. Then, five cards are placed face up on the table. Each player can then use any three of these five cards along with the two cards they were dealt to form the best poker hand possible. The "least likely" of any poker hands (as you determined in Part I) is considered the "best" poker hand.

Suppose that Phil was dealt the King of Hearts (K♡) and the King of Diamonds (K♢) and Chris was dealt the Ace of Spades (A♠) and the King of Clubs (K♣). Before any of the five cards are placed face up on the table, determine the probability that Chris will win the hand.

Alternate Version of Part III:
Instead of analyzing one particular hand of Texas Hold 'Em as above, write a computer program that computes the probabilities based on any initial hands of Texas Hold 'Em. The program should accept as input the two cards that each player was dealt and return the probability that each person has of winning the hand prior to any cards being placed face up on the table. If you decide to do this version of Part III, I would be more than happy to give you some ideas as to how to get started.

4.2 Yahtzee

Part I:
Consider the experiment of rolling five dice all at once. Clearly define the 6 different possible results (Three of a kind, Four of a kind, Full House, Small Straight, Large Straight and YAHTZEE). For each result, explain how to calculate the number of throws that yield that result. Summarize your results in a table, including the probability of each result. To help with your calculations, you may assume that the five dice are each colored a different color so that they are distinguishable.

Part II:
Now suppose that after you roll five dice, you are allowed to select any of the five dice and roll them again. At which point, you may select any of the five dice and roll them for a third time. For this part, we will calculate the probability of rolling YAHTZEE after any of the three rolls.

First, consider the situation where you have i dice of the same value and you re-roll the other $5-i$ dice. Let $p_{i,j}$ denote the probability that you now have exactly j dice all of the same value. Note that if $j < i$ then $p_{i,j} = 0$ since you certainly have at least i dice all of the same value after you roll the $5-i$ other dice.

Furthermore, notice that the j dice you end up with may not have the same value of the i dice you started with. For example, you could start with five dice with values $(2, 2, 3, 4, 5)$. Now suppose that you re-roll the three dice with values $(3, 4, 5)$ and get three dice all of value 1. In other words, your five dice now have the values $(1, 1, 1, 2, 2)$. This situation would be accounted for in calculating $p_{2,3}$.

Calculate all of the entries in the following matrix T, which is called a *transition matrix*.

$$T = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} & p_{1,5} \\ 0 & p_{2,2} & p_{2,3} & p_{2,4} & p_{2,5} \\ 0 & 0 & p_{3,3} & p_{3,4} & p_{3,5} \\ 0 & 0 & 0 & p_{4,4} & p_{4,5} \\ 0 & 0 & 0 & 0 & p_{5,5} \end{bmatrix}$$

Give a detailed explanation of how to calculate each of the values $p_{1,1}$, $p_{1,2}$, $p_{1,3}$, $p_{1,4}$ and $p_{1,5}$. Once you have calculated all of the entries in the above matrix, let me know and I will help you calculate T^3. The entry in the first row and fifth column of T^3 is the probability of getting YAHTZEE using at most three rolls.

Part III:
The upper portion of a typical YAHTZEE scorecard has a bonus of 35 points if a minimum score is reached. In particular, there are six places to record the total (after three rolls) of all the ones, twos, threes, fours, fives, or sixes. For example, if you roll three 1's and two 5's, you could score 3 points for the 1's or 10 points for the 5's. If the total of these six recorded values is at least 63, then you get a bonus score of 35 points.

Calculate the number of ways that you can achieve this bonus. In other words, if x_i is the number of i's after your three rolls, then you would record $i \times x_i$ and thus your total would be $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$. Therefore, we are in search of the number of nonnegative integer solutions to

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \geq 63.$$

To find this number, first find a generating function for the number of nonnegative integer solutions to

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 = n.$$

Once you have done this, let me know and I will help you calculate the number of nonnegative integer solutions to $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \geq 63$.

4.3 PowerBall

Part I:
Consider the experiment of playing a single game of PowerBall. The rules of the game, 9 different ways to win, dollar value of each prize and the odds of winning are provided by www.powerball.com. Explain how to calculate the probability of winning each of the nine different prizes. Summarize your results in a table.

Part II:
The amount of money that you would expect to win from a single game of PowerBall is calculated using the following formula

$$\sum_{i=1}^{9} w_i P(A_i)$$

where $A_1, A_2, \ldots, A_9$ represent the nine different ways to win and w_i is the amount of money that you would win if the outcome is in event A_i.

Calculate the amount of money you would expect to win from a single game of Powerball, using the all time record Grand Prize jackpot of 365 million dollars. How large does the PowerBall jackpot have to be in order for it to be worth your while to play? In other words, how large would the jackpot have to be in order for you to expect to at least win back the one dollar you paid to play the game in the first place?

For an additional dollar, you can play PowerPlay. That is, during each drawing of the PowerBall numbers, a PowerPlay number is also selected. This number is a 2, 3, 4 or 5 with each number being equally likely. Any dollar amount you win from PowerBall (except the Grand Prize) is multiplied by this PowerPlay number. Calculate the amount of money that you would expect to win from playing a single game of PowerPlay. Is it worth the extra dollar to play PowerPlay?

Part III:

Analyze any of the Pennsylvania State Lotteries (see www.palottery.com) or another State Lottery (google "state lottery"). Explain how to calculate the odds of winning and the amount of money that you would expect to win. If you decide to do this particular project, let me know which State Lottery you decide to analyze so that I may approve it.