July 25, 2006 Math 035 Assignment #6 Solutions

p. 48-50, #1, 8, 16, 20, 21

1. (a) List all the divisors of 100.
(b) List all the divisors of 75.
(c) List all the common divisors of 75 and 100.
(d) What is (75, 100)?

Solution:
(a) The divisors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, and 100.
(b) The divisors of 75 are 1, 3, 5, 15, 25, and 75.
(c) The common divisors of 75 and 100 are 1, 5, and 25.
(d) The GCD is $(75, 100) = 25$.

8. Using the technique given in this chapter, find a pair of integers M and N such that
(a) $(24, 51) = M \cdot 24 + N \cdot 51$
(b) $(57, 133) = M \cdot 57 + N \cdot 133$
(c) $(13, 21) = M \cdot 13 + N \cdot 21$

Solution:
(a) First apply the Euclidean algorithm:

$$\begin{aligned} 51 &= 2 \cdot 24 + 3 \\ 24 &= 8 \cdot 3 + 0 \end{aligned}$$

Thus $(24, 51) = 3$, and by subtracting $2 \cdot 24$ from both sides of the first equation, we get $3 = 51 - 2 \cdot 24$, so $M = -2$, $N = 1$.
(b) Just as above,

$$\begin{aligned} 133 &= 2 \cdot 57 + 19 \\ 57 &= 3 \cdot 19 + 0 \end{aligned}$$

So $(57, 133) = 19$, and the first equation gives $19 = 133 - 2 \cdot 57$. Thus $M = -2$, $N = 1$ again.

(c) This one is a little bit longer:

$$\begin{aligned}
21 &= 1 \cdot 13 + 8 \\
13 &= 1 \cdot 8 + 5 \\
8 &= 1 \cdot 5 + 3 \\
5 &= 1 \cdot 3 + 2 \\
3 &= 1 \cdot 2 + 1 \\
2 &= 2 \cdot 1 + 0
\end{aligned}$$

So $(21, 13) = 1$, and the second to last equation in the list gives

$$1 = 3 - 1 \cdot 2$$

The fourth equation tells us that $2 = 5 - 1 \cdot 3$, and substituting this into the above equation gives

$$\begin{aligned}
1 &= 3 - 1 \cdot 2 \\
&= 3 - 1 \cdot (5 - 1 \cdot 3) \\
&= 3 - 1 \cdot 5 + 1 \cdot 3 \\
&= 2 \cdot 3 - 1 \cdot 5
\end{aligned}$$

The third equation gives $3 = 8 - 1 \cdot 5$, and putting this in the equation we just derived gives

$$\begin{aligned}
1 &= 2 \cdot 3 - 1 \cdot 5 \\
&= 2 \cdot (8 - 1 \cdot 5) - 1 \cdot 5 \\
&= 2 \cdot 8 - 2 \cdot 5 - 1 \cdot 5 \\
&= 2 \cdot 8 - 3 \cdot 5
\end{aligned}$$

The second equation lets us replace 5 and 8 by 8 and 13:

$$\begin{aligned}
5 &= 13 - 1 \cdot 8 \\
1 &= 2 \cdot 8 - 3 \cdot 5 \\
&= 2 \cdot 8 - 3 \cdot (13 - 1 \cdot 8) \\
&= 2 \cdot 8 - 3 \cdot 13 + 3 \cdot 8 \\
&= 5 \cdot 8 - 3 \cdot 13
\end{aligned}$$

Finally, the first equation takes us all the way back to 13 and 21:

$$\begin{aligned} 8 &= 21 - 1 \cdot 13 \\ 1 &= 5 \cdot 8 - 3 \cdot 13 \\ &= 5 \cdot (21 - 1 \cdot 13) - 3 \cdot 13 \\ &= 5 \cdot 21 - 5 \cdot 13 - 3 \cdot 13 \\ &= 5 \cdot 21 - 8 \cdot 13 \end{aligned}$$

so we can take $M = -8$ and $N = 5$.

16. Find (96, 144) in three ways:
(a) by listing all the divisors of 96 and of 144 and finding which is the greatest common divisor
(b) by the Euclidean Algorithm
(c) by expressing 96 and 144 as the product of primes and using the Fundamental Theorem of Arithmetic.

Solution:
(a) The divisors of 96 are 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, and 96; the divisors of 144 are 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, and 144. So the common divisors are 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48. The GCD is 48.
(b)

$$\begin{aligned} 144 &= 1 \cdot 96 + 48 \\ 96 &= 2 \cdot 48 + 0 \end{aligned}$$

So the GCD is 48.
(c) $96 = 2^5 \cdot 3$, and $144 = 12^2 = 2^4 \cdot 3^2$. Since taking the GCD using the FTA consists of selecting the smallest exponenet from each prime, the GCD of 96 and 144 is $2^4 \cdot 3$, which works out to 48.

20. Using the FTA, find (a) $\mathrm{GCD}(2^5, 3^7)$ and (b) $\mathrm{GCD}(2^4 \cdot 5^3, 2^2 \cdot 5 \cdot 7)$.

Solution:
(a) The GCD is 1, since the two numbers given share no prime factor.

(b) The GCD is $2^2 \cdot 5 = 20$; there can be no factor of 7, since the first of the two numbers is not a multiple of 7.

21. Let LCM denote the Least Common Multiple of two numbers, and find
(a) $\mathrm{LCM}(4, 7)$
(b) $\mathrm{LCM}(4, 6)$
(c) $\mathrm{LCM}(2^5 \cdot 3^2, 2^4 \cdot 3^6)$
(d) $\mathrm{LCM}(144, 96)$

Solution:
(a) $\mathrm{LCM}(4, 7) = \mathrm{LCM}(2^2, 7) = 2^2 \cdot 7 = 28$
(b) $\mathrm{LCM}(4, 6) = \mathrm{LCM}(2^2, 2 \cdot 3) = 2^2 \cdot 3 = 12$
(c) $\mathrm{LCM}(2^5 \cdot 3^2, 2^4 \cdot 3^6) = 2^5 \cdot 3^6$, by taking the maximum exponent from each prime.
(d) $\mathrm{LCM}(144, 96) = \mathrm{LCM}(2^4 \cdot 3^2, 2^5 \cdot 3) = 2^5 \cdot 3^2$.