July 21, 2006 Math 035 Assignment #5 Solutions

p. 27-30, #9, 10, 18, 26, 41

9. If you feed the Prime-manufacturing Machine 11 and 13, what primes does it produce?

Solution: The machine multiplies 11 and 13 to get $A = 143$, then adds one to get $B = 144$. It returns the prime factors of B, and since $B = 12 \cdot 12 = 2^2 \cdot 3 \cdot 2^2 \cdot 3 = 2^4 \cdot 3^2$, the output consists of the prime numbers 2 and 3.

10. If you feed the Prime-manufacturing Machine 2, 11, and 13, what primes does it produce?

Solution: This time the product obtained is $A = 2 \cdot 11 \cdot 13 = 2 \cdot 143 = 286$, and adding one gives $B = 287$. Checking divisors reveals that this is not a multiple of 3, nor of 5, but that 7 is a divisor, so $B = 7 \cdot 41$, and the output is the two prime numbers 7 and 41.

18. Let N be a natural number larger than 1, and let D be the smallest divisor of N other than 1. Explain why D is prime.

Solution: Any divisor of D is also a divisor of N - this is one of the basic properties of divisibility. If A divides D, then it also divides any multiple of D, including N. So if D had a divisor A apart from 1 and itself, A would also be a divisor of N, but A is smaller than D, which is the smallest such divisor. So no such A exists, meaning that the only divisors of D are 1 and itself, meaning that D is prime.

26. Is it true that whenever 10 divides the product of two natural numbers it must divide at least one of them?

Solution: No. For example, 10 divides $2 \cdot 5 = 10$, but it does not divide either 2 or 5. Other examples are numerous; in fact, any multiple of 10 can be factored in such a way that 10 does not divide either of the factors. $20 = 4 \cdot 5$, $30 = 6 \cdot 5$, $40 = 8 \cdot 5$, $50 = 2 \cdot 25$, and so on.

41. To determine whether 197 is prime why does it suffice to check whether any of the primes 2, 3, 5, 7, 11, 13 divide 197?

Solution: There are really two questions to be answered. First, why do we only need to check primes (why do we not need to check 4, 6, 8, etc.)? Second, why can we stop at 13?

We only need to check prime numbers since every composite number has a prime factor; if 197 is not prime, then not only does it have some divisor between 1 and itself, but it has a *prime* divisor between one and itself. For example, the smallest such divisor will be prime (this was question 18). So we do not need to check 4, 6, 8, etc. - if 4 divides the number we are testing, then so does 2, so we will discover it is not prime when we check 2. If 9 divides it, we will catch that on 3, and so on.

We can stop at 13 because any composite number has a factor (besides 1) which is no bigger than its square root. When we list all the divisors of a number, we wind up with two columns; for example, all the divisors of 60 are as listed below:

1	60
2	30
3	20
4	15
5	12
6	10

The left-hand column contains all the divisors which are less than the square root of the number (in this case, $\sqrt{60}$ is about 7.75); the right-hand column contains all the divisors which are greater than the square root. So if 197 has any factors besides 1 and itself, one of them will appear in the left-hand column, and will be no more than $\sqrt{197}$, which is about 14; thus we need only check the prime numbers up to 14, which are 2, 3, 5, 7, 11, and 13.

Another way of thinking of this is that each row of the table contains two numbers; the product of the two numbers in any row is 60, the number we started with. But if we factor N (in this case, $N = 197$) into two numbers A and B, so $N = AB$, then one of those two must be less than the square root of N; if both are greater than that square root, we have $N = AB > \sqrt{N} \cdot \sqrt{N} = N$, but N cannot be greater than itself. So if N is composite, it has a prime factor less than its square root.

Note that rather than calculating the square root of N, you could also look at all the prime numbers whose square is less than or equal to N; it works out to the same thing. So in this case, $13^2 = 169$ is less than 197, but $17^2 = 289$ is too big, so we only need to check the primes up to 13.