

Generating E_0 by Homeomorphisms

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Definition 1. The equivalence relation E_0 is defined on the Cantor space 2^ω by

$$xE_0y \text{ iff } \{n : x(n) \neq y(n)\} \text{ is finite.}$$

- E_0 is naturally generated by the action of the group $\mathbb{Z}_2^{<\omega}$ (which we can identify with $2^{<\omega}$) given by $s \cdot x = x \Delta s$.
- E_0 is *almost* generated by the *odometer* map φ_0 given by

$$\begin{aligned} \varphi_0(1^n \frown 0 \frown \alpha) &= 0^n \frown 1 \frown \alpha \\ \varphi_0(1^\infty) &= 0^\infty \end{aligned}$$

i.e. “adding 1 mod 2 with right carry”.

- The orbits of φ_0 are the E_0 equivalence classes, with the exception that the two E_0 classes of 0^∞ and 1^∞ are combined into one orbit.

- In many categories, such as measure-preserving transformations or Borel automorphisms, this difference is inessential since we can easily modify φ_0 to fix this problem.
- In a topological setting, though, this becomes non-trivial.
- The odometer has many nice properties: It is a homeomorphism, in fact a Lipschitz automorphism, in fact generated by a finite automaton, shift-similar (defined below), and others.
- We will consider which of these properties can be retained by a function which literally generates E_0 .

We first consider whether a homeomorphism can generate E_0 .

The following theorem of Giordano, Putnam, and Skau implies that it can be.

Theorem 2 (Giordano, Putnam, Skau). *Every AF equivalence relation is isomorphic to the orbit equivalence relation induced by the Vershik automorphism associated to some Bratelli diagram.*

An AF equivalence relation is one which is the increasing union of compact equivalence relations, where each relation is relatively open in the next. E_0 is an AF equivalence relation.

We will give here an explicit construction of a homeomorphism which generates E_0 .

Theorem 3. *The equivalence relation E_0 is the orbit equivalence relation of a homeomorphism of 2^ω .*

Proof: Consider the Bratelli diagram given below.

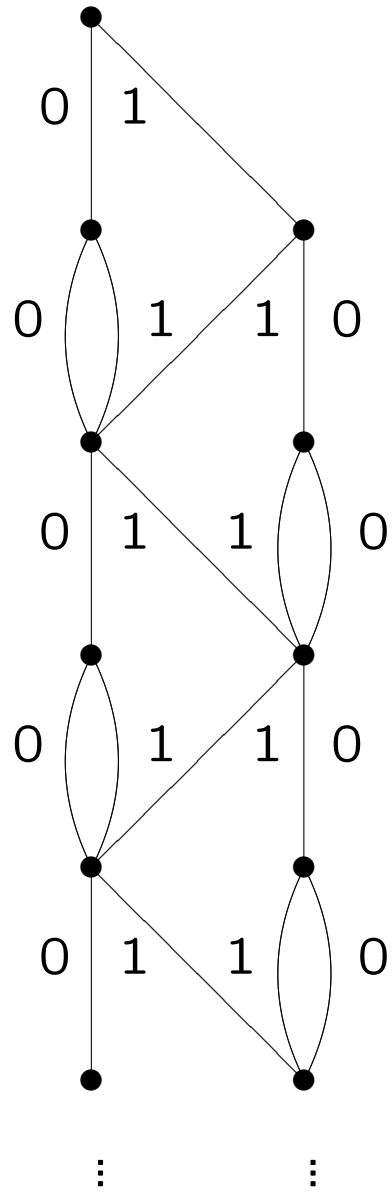
Let φ be the Vershik automorphism associated to the diagram, i.e. φ sends a path to the lexicographically next path (where edges are ordered from left-to-right).

The unique maximal path $10000\dots$ is sent to the unique minimal path $00000\dots$.

Labels along paths indicate coordinates in 2^ω .

The diagram is *simple*, so the orbit equivalence relation of φ is tail equivalence of paths, together with the connection between the maximal and minimal paths.

Then φ induces E_0 . □



The Bratelli diagram for φ

Other Groups

Since E_0 is isomorphic to $E_0 \times E_0$ via a homeomorphism, E_0 can be generated by two commuting homeomorphisms, where one acts on the even coordinates and the other acts on the odd coordinates. Each of these homeomorphisms will then have nowhere dense orbits.

Question 4. *Can E_0 be generated by two commuting homeomorphisms with dense orbits?*

This is related to a topological version of the result that any Borel equivalence relation generated by an action of \mathbb{Z}^2 is hyperfinite.

Question 5. *Which other groups can generate E_0 via homeomorphisms? Which can generate E_0 via homeomorphisms with dense orbits?*

Nicer Maps

We can strengthen the notion of homeomorphism.

Definition 6. A *Lipschitz automorphism* of 2^ω is a homeomorphism such that if $x \upharpoonright n = y \upharpoonright n$, then $\varphi(x) \upharpoonright n = \varphi(y) \upharpoonright n$.

Question 7. *Can E_0 be generated by a Lipschitz automorphism?*

We suspect that the answer is yes, but the combinatorics must be non-trivial as results below will show.

A particularly simple type of Lipschitz automorphism is one generated by a finite automaton.

Question 8. *Can E_0 be generated by a finite automaton?*

Again, we do not know the answer to this.

We can rule out certain niceness properties of a homeomorphism generating E_0 .

Definition 9. If φ generates E_0 , we define the $(\mathbb{Z}_2^{<\omega}, \mathbb{Z})$ -cocycle to be the map $n : \mathbb{Z}_2^{<\omega} \times 2^\omega \rightarrow \mathbb{Z}$ given by

$$n(s, x) = \text{the unique } k \in \mathbb{Z} \text{ such that} \\ x \Delta s = \varphi^k(x).$$

We say the cocycle is continuous if it is continuous as a function where $\mathbb{Z}_2^{<\omega}$ and \mathbb{Z} are given the discrete topology,

This definition makes sense since if we just require that φ generates an equivalence relation containing E_0 . In this setting, the cocycle for the odometer is continuous.

Proposition 10. *There is no homeomorphism which generates E_0 such that the $(\mathbb{Z}_2^{<\omega}, \mathbb{Z})$ -cocycle is continuous.*

Proof: If the cocycle is continuous, then each of the sets $\{x : n(s, x) = k\}$ with s and k fixed is clopen. Let $s \neq \emptyset$, so $n(s, x) \neq 0$ for all x . Then the sets

$$\begin{aligned}\text{Pos}_s &= \{x : n(s, x) > 0\} \\ \text{Neg}_s &= \{x : n(s, x) < 0\}\end{aligned}$$

are both open; since their union is 2^ω they are in fact clopen. Note that the collection $\{\text{Pos}_s : s \neq \emptyset\}$ has the finite intersection property, since if we take the finite subgroup generated by a finite collection of s 's and look at the (finite) orbit of some point x , the element of this orbit which occurs first in the φ -orbit of x will be in the intersection. Hence, by compactness, the set

$$\bigcap_{s \neq \emptyset} \text{Pos}_s \neq \emptyset.$$

But now if $x \in \bigcap_{s \neq \emptyset} \text{Pos}_s$, then for each $s \neq \emptyset$ we have that $x\Delta s$ is in the positive φ -orbit of x . So any point in the negative φ -orbit of x must differ from x on infinitely many coordinates.

□

Definition 11. We say a homeomorphism φ is *shift-similar* if for all x we have $\varphi^2(x) = x(0) \frown \varphi(x^*)$, where $x^*(n) = x(n + 1)$.

Note that the odometer is shift-similar.

Proposition 12. *There is no shift-similar homeomorphism which generates E_0 .*

Proof: First we show that a shift-similar homeomorphism with dense orbits must be a Lipschitz automorphism (In fact any shift-similar homeomorphism is Lipschitz).

Define the sets

$$\mathcal{F}_n = \{x : \varphi(x)(n) \neq x(n)\}.$$

Each \mathcal{F}_n is clopen. We will show that \mathcal{F}_n is φ^{2^n} -invariant. Since the φ^{2^n} -orbit of x is dense in the neighborhood $\{y : y \upharpoonright n = x \upharpoonright n\}$ (being the E_0 class of x in that interval), this shows that each \mathcal{F}_n is the union of clopen sets determined by sequences of length n .

Hence x 's membership in \mathcal{F}_n is determined by $x \upharpoonright n$. Thus, so is $\varphi(x) \upharpoonright n$, hence φ is Lipschitz.

To see this, note that we have

$$\varphi^{2^n}(x) = x(0) \wedge \cdots \wedge x(n-1) \wedge \varphi(x^{*n}).$$

First, since $\varphi^2(x)(0) = x(0)$, we have that $x \in \mathcal{F}_0$ if and only if $\varphi(x) \in \mathcal{F}_0$, so \mathcal{F}_0 is φ -invariant. Since φ has dense orbits, it must in fact be all of 2^ω . Using shift-similarity, we similarly have that for all x , $\varphi^{2^n}(x)(n) \neq x(n)$. Hence, for each x ,

$$\left| \{k : 0 \leq k < 2^n \text{ and } \varphi^k(x) \in \mathcal{F}_n\} \right| \text{ is odd.}$$

Replacing x with $\varphi(x)$,

$$\left| \{k : 0 < k \leq 2^n \text{ and } \varphi^k(x) \in \mathcal{F}_n\} \right| \text{ is odd.}$$

Hence $x \in \mathcal{F}_n$ iff $\varphi^{2^n}(x) \in \mathcal{F}_n$.

We now show that any shift-similar Lipschitz automorphism with dense orbits must change all digits of some point.

We will actually establish the following characterization: A non-identity Lipschitz automorphism is shift-similar if and only if there is a sequence $\langle k_n \rangle_{n \in \omega}$ such that for all n , $\varphi \upharpoonright n = \varphi_0^{k_n} \upharpoonright n$, i.e. φ is the limit of a sequence of powers of the odometer.

These turn out to be precisely the homeomorphisms which commute with the odometer.

Each of these can also be viewed as an automorphism of the form $\varphi(x) = x + \alpha$, where $\alpha \in 2^\omega$ and addition is mod 2 with right carry.

Hence, if φ has dense orbits (so $\alpha(0) = 1$), and we let x be the sequence with $x(n) = 1 - \alpha(n + 1)$ for all n , then $\varphi(x)(n) = 1 - x(n)$ for all n .

Each such automorphism is shift-similar.

For the converse, it suffices to establish by induction on n that for φ shift-similar there is a k such that $\varphi \upharpoonright n = \varphi_0^k \upharpoonright n$. The case $n = 0$ is trivial. So suppose k is such that $\varphi \upharpoonright n = \varphi_0^k \upharpoonright n$ and consider $\varphi \upharpoonright (n + 1)$. For $s \in 2^{n+1}$ we have that

$$\begin{aligned} \varphi(\varphi(s)) &= s(0) \frown \varphi(s^*) \\ &= s(0) \frown \varphi_0^k(s^*) = \varphi_0^{2k}(s). \end{aligned}$$

Hence, $\varphi \upharpoonright (n + 1)$ is completely determined from $\varphi \upharpoonright n$ and $\varphi(0^{n+1})$ since the orbits are dense. Thus, there are at most two distinct possibilities for $\varphi \upharpoonright (n + 1)$. Since φ_0^k and $\varphi_0^{k+2^n}$ are distinct and satisfy the given conditions, $\varphi \upharpoonright (n + 1)$ must be one of these and we are done. \square

We have actually established:

Corollary 13. *A homeomorphism is shift-similar if and only if it is in the commutant of the odometer.*

Searching for shift-similar homeomorphisms to generate E_0 would have been nice, since we only need to verify that $f_0(x)$ is in the orbit of x , where $f_0(x)$ flips the first coordinate of x .

A weakening of shift-similarity would actually have the necessary combinatorial properties.

Definition 14. We say a homeomorphism φ is *left shift-similar* if for all y we have $\varphi^2(0 \frown y) = 0 \frown \varphi(y)$. Right shift-similarity is defined similarly, with 1 in place of 0.

Question 15. *Can E_0 be generated by a left shift-similar homeomorphism?*